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GRAPHS WHOSE WEAK ROMAN DOMINATION NUMBER INCREASES BY THE REMOVAL OF ANY EDGE

Rihab Hamid, Nour Elhouda Bendahib, Mustapha Chellali and Nacéra Meddah

ABSTRACT

Let $f : V \to \{0,1,2\}$ be a function on a graph $G = (V,E)$. A vertex v with $f(v) = 0$ is said to be undefended with respect to f if it is not adjacent to a vertex u with $f(u) > 0$. A function f is called a weak Roman dominating function (WRDF) if each vertex v with $f(v) = 0$ is adjacent to a vertex u with $f(u) > 0$, such that the function f' defined by $f'(v) = 1$, $f'(u) = f(u)-1$, and $f'(w) = f(w)$ for all $w \in V \setminus \{v,u\}$, has no undefended vertex. The weight of a WRDF is the sum of its function values over all vertices, and the weak Roman domination number $\gamma_r(G)$ is the minimum weight of a WRDF in G. In this paper, we consider the effects of edge deletion on the weak Roman domination number of a graph. We show that the deletion of an edge of G can increase the weak Roman domination number by at most 1. Then we give a necessary condition for γ_r-ER-critical graphs, that is, graphs G whose weak Roman domination number increases by the deletion of any edge. Restricted to the class of trees, we provide a constructive characterization of all γ_r-ER-critical trees.
Keywords : Weak Roman domination, weak Roman domination critical graphs, trees.

1. INTRODUCTION

Let G be a simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *open neighborhood* of a vertex $v \in V(G)$ is $N(v) = \{u \in V(G) \mid uv \in E(G)\}$, and its *closed neighborhood* $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex v is $\deg_G(v) = |N(v)|$. A vertex of degree one is called a *leaf*, and its neighbor is called a *stem*. A stem is said to be *strong* if it is adjacent to at least two leaves. Let $S(G)$ denote the set of stems of G. An edge incident to a leaf is called a *pendant edge*. A *tree* is an acyclic connected graph. A tree T is a *double star* if it contains exactly two vertices that are not leaves. A *subdivided star* SS_t is a tree obtained from a star $K_{1,t}$ by replacing each edge uv of $K_{1,t}$ by a vertex w and edges uw and vw, while a *double subdivided star* SS_t^* is obtained from $K_{1,t}$ by replacing each edge uv of $K_{1,t}$ by two vertices x,y and edges ux,xy and yv. For a vertex v in a rooted tree T, we denote by $C(v)$ and $D(v)$ the set of children and descendants, respectively, of v. The *maximal subtree* at v is the subtree of T induced by $D(v) \cup \{v\}$, and is denoted by T_v.

Introduced in 2004 by Cockayne, Dreyer, Hedetniemi and Hedetniemi [4], the concept of Roman domination is now well-studied with over 200 papers published on it and its variations. For more on Roman domination, we refer the reader to the book chapters and survey in [1, 2, 3].

In this paper, we are interested in weak Roman domination, a less restrictive version of Roman domination, introduced by Henning and Hedetniemi [5]. For a graph G, let $f : V(G) \to \{0,1,2\}$ be a function. If $V_i = \{v \in V | f(v) = i\}$ for $i \in \{0,1,2\}$, then f can be denoted by $f = (V_0,V_1,V_2)$. A vertex v with $f(v) = 0$ is said to be *undefended* with respect to f if it is not adjacent to a vertex u with $f(u) > 0$. A function f is called a *weak Roman dominating function* (WRDF) if each vertex v with $f(v) = 0$ is adjacent to a vertex u with $f(u) > 0$, such that the function $f' = (V_0',V_1',V_2')$ defined by $f'(v) = 1$, $f'(u) = f(u)-1$, and $f'(w) = f(w)$ for all $w \in V \setminus \{v,u\}$, has no undefended vertex. In this case, we will say that vertex u is a *moving neighbor* for v. Note that every vertex in V_0 has at least one moving neighbor in $V_1 \cup V_2$.

The *weight* of a WRDF is the value $f(V) = \sum_{u \in V(G)} f(u)$, and the *weak Roman domination number* $\gamma_r(G)$ is the minimum weight of a WRDF in G. A WRDF f is called a $\gamma_r(G)$-function if $f(V) = \gamma_r(G)$.

In this paper, we study the effects of edge removal on the weak Roman domination number of a graph. In particular, we consider graphs that are "critical" in the sense that the weak Roman domination number increases upon the removal of an arbitrary edge. Before presenting our main results, let us provide some useful definitions and remark.

Definition 1 *If $f = (V_0, V_1, V_2)$ is a $\gamma_r(G)$-function, then for every vertex $v \in V_1 \cup V_2$, let $P_f(v)$ be the set of all vertices in $N(v) \cap V_0$ for which v is a moving neighbor.*

Definition 2 *A vertex $x \in V(G)$ is said to be good, if there exists a $\gamma_r(G)$-function f such that $f(x) > 0$. Moreover, a good vertex $x \in V(G)$ is 1-good, if no $\gamma_r(G)$-function assigns a 2 to x.*

Remark 1 *For every graph G with $S(G) \neq \emptyset$, there exists a $\gamma_r(G)$-function $f = (V_0, V_1, V_2)$ such that $S(G) \subseteq V_1 \cup V_2$.*

2. EDGE REMOVAL

We begin this section by showing that the removal of an edge in G cannot decrease the weak Roman domination number but can increase it by at most one.

Proposition 1 *For a graph G and edge $uv \in E(G)$, $\gamma_r(G) \leq \gamma_r(G - uv) \leq \gamma_r(G) + 1$.*

To see the sharpness of the bounds in Proposition 1, we simply consider the example of a path $P_4 = abcd$, where $\gamma_r(P_4 - bc) = \gamma_r(P_4) = 2$ while $\gamma_r(P_4 - ab) = \gamma_r(P_4) + 1 = 3$.

According to Proposition 1, an edge $e \in E(G)$ is said to be *critical* if $\gamma_r(G - uv) = \gamma_r(G) + 1$. Moreover, a graph G is called *γ_r-ER-critical* if $\gamma_r(G - e) = \gamma_r(G) + 1$ for all $e \in E(G)$, that is, every edge of G is critical.

Our next result gives a necessary condition for γ_r-*ER*-critical graphs.

Proposition 2 *Let G be γ_r-ER-critical graph. Then for every $\gamma_r(G)$-function $f = (V_0, V_1, V_2)$ we have :*

(1) $V_1 \cup V_2$ is an independent set.
(2) If $v \in V_2$, then for every $x \in P_f(v)$, $N(x) \cap V_0 = \emptyset$. In particular $P_f(v)$ is independent.
(3) For every $u, v \in V_1 \cup V_2$, no edge joins $P_f(v)$ and $P_f(u)$.
(4) For every vertex $v \in V_0$, $|N(v) \cap (V_1 \cup V_2)| \leq 2$. In particular, $|N(v) \cap V_1| \leq 2$, $|N(v) \cap V_2| \leq 1$ and if $N(v) \cap V_2 \neq \emptyset$, then $N(v) \cap V_1 = \emptyset$.
(5) If G is connected, then $|V_2| \leq 1$.

We note that the converse of Proposition 2 is not true. To see consider the graph G shown in Figure 1, where G has a unique $\gamma_r(G)$-function f that assigns a 1 to dark vertices of G and a 0 to the remaining vertices. Clearly f satisfies all conditions of Proposition 2, but $\gamma_r(G - uv) = \gamma_r(G) = 5$.

The following corollary is an immediate consequence of Remark 1 and Proposition 2.

Corollary 3 *If a graph G is a γ_r-ER-critical, then the set of stems is independent.*

Proposition 4 *If G is a connected γ_r-ER-critical graph, then there is a $\gamma_r(G)$-function $f = (V_0, V_1, V_2)$ such that $|V_2| = 1$ if and only if G is a star of order at least 4.*

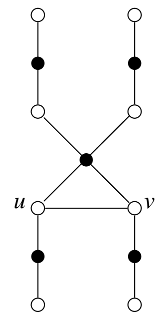

FIGURE 1 – Graph G

3. γ_R-ER-CRITICAL TREES

In the aim to characterize γ_r-ER-critical trees, let $\mathscr{H}$ be the collection of all trees that can be obtained from a sequence $T_1, T_2, ..., T_p$, $(p \geq 1)$, of trees T such that T_1 is the path P_2, and if $p \geq 2$, then T_{i+1} can be obtained recursively from T_i by one of the two operations $\mathscr{O}_1$ and $\mathscr{O}_2$ defined below. Let one the vertices of T_1 be considered a stem and the other a leaf. Let also H_t be the tree obtained from a double subdivided star SS_t^*, with $t \geq 2$, centered at y by adding a new vertex z and the edge yz (for example, see the tree H_3 shown in Figure 2). It is worth noting that $d_{H_t}(y) = t+1$ and $\gamma_r(H) = t+2$. Recall that a *pendant path* P of a graph G is an induced path such that one of the endvertices is a leaf in G, and its other endvertex is the only vertex of P adjacent to some vertex in $G - P$.

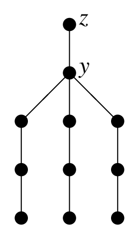

FIGURE 2 – Tree H_3

- **Operation** $\mathscr{O}_1$ **:** Assume x is a stem of T_i. Then T_{i+1} is obtained from T_i by adding a path $P_5 = x_1x_2x_3x_4x_5$ attached by an edge x_5x.
- **Operation** $\mathscr{O}_2$: Assume u is vertex of T_i which is either a stem or belongs to a pendant path of length 3. Then T_{i+1} is obtained from T_i by adding a copy of H_t attached by an edge uz.

Our main result is the following.

Theorem 5 *A non-trivial tree T is γ_r-ER-critical if and only if T is a star of order at least four or $T \in \mathscr{H}$.*

4. CONCLUSIONS

In this paper, we showed that the delection of an edge of a graph G can increase the weak Roman domination number by at most 1. Then we established a necessary condition for graphs whose number of Weak Roman domination number increases by the removal of any edge. Finally, we provided a constructive characterization of all γ_r-ER-critical trees. We close by mentioning that it is also worthwhile to characterize other γ_r-ER-critical graphs such as cactus graphs.

5. REFERENCES

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