THE EFFECT OF EDGE LIFTING ON ROMAN DOMINATION IN GRAPHS

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ABSTRACT

Let $G$ be a graph, and let $uxv$ be an induced path centered at $x$. An edge lift defined on $uxv$ is the action of removing edges $ux$ and $vx$ while adding the edge $uv$ to the edge set of $G$. In this paper, we initiate the study of the effects of edge lifting on the Roman domination number of a graph, where various properties are established. A characterization of all trees for which every edge lift increases the Roman domination number is provided. Moreover, we characterize the edge lift of a graph decreasing the Roman domination number, and we show that there are no graphs with at most one cycle for which every possible edge lift can have this property.

1. INTRODUCTION

We consider simple graphs $G$ with vertex set $V = V(G)$ and edge set $E = E(G)$. The open neighborhood of a vertex $v \in V$ is the set $N_{G}(v) = N(v) = \{u \in V \mid uv \in E\}$, the closed neighborhood of $v$ is the set $N_{G}[v] = N[v] = N_{G}(v) \cup \{v\}$, and the degree of $v$ is $\deg_{G}(v) = |N_{G}(v)|$. A vertex of degree one is called a leaf and its neighbor a support vertex. If $v$ is a support vertex, then $L_{v}$ will denote the set of the leaves attached at $v$. An S-external private neighbor of a vertex $v \in S$ is a vertex $u \in V \setminus S$ which is adjacent to $v$ but to no other vertex of $S$. The set of all $S$-external private neighbors of $v \in S$ is called the $S$-external private neighbor set of $v$ and is denoted $\text{epn}(v,S)$. Let $uxv$ be an induced path in a graph $G$. We say that $ux$ and $vx$ are lifted off $x$ and call the operation of removing $ux$ and $vx$ and adding $uv$ to $E(G)$ an edge lift. Moreover, the graph formed by $G$ by an edge lift on $uxv$ is denoted as $G_{uv}$, where $V(G_{uv}) = V(G)$ and $E(G_{uv}) = \{uv\} \cup E(G) \setminus \{ux, vx\}$. The process of edge lifting (also called edge splitting) was introduced by Lovász in [6, 7] to study edge connectivity in graphs, and has been extended in 2011 to domination in graphs (see [3, 4]) and in 2013 to total domination in graphs (see [5]) to study its effects on the corresponding domination parameters. Our aim is to study the effect of edge lifting with respect to Roman domination. A function $f : V \rightarrow \{0, 1, 2\}$ is a Roman dominating function, abbreviated RDF, on a graph $G$ if for every vertex $v \in V$ with $f(v) = 0$, there exists a neighbor $u \in N(v)$ with $f(u) = 2$. The weight of an RDF $f$ is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of an RDF on $G$ is called the Roman domination number of $G$ and is denoted by $\gamma_{R}(G)$. An RDF on $G$ with weight $\gamma_{R}(G)$ is called a $\gamma_{R}(G)$-function. Notice that an RDF $f$ on $G$ can be denoted by $(V_{0}, V_{1}, V_{2})$ (or $(V_{0}^{j}, V_{1}^{j}, V_{2}^{j})$ to refer to $f$), where $V_{j} = \{v \in V \mid f(v) = j\}$ for $j \in \{0, 1, 2\}$. The Roman domination number was introduced by Cockayne et al. [2] in 2004 and was inspired by the work of ReVelle and Rosing [9] and Stewart [10]. For more information on Roman domination, we refer the reader to the book chapter [11]. All graphs considered are connected and contain at least one induced path on three vertices. For every graph $G$ we define the following weak partition (in which a subset may be
empty) $A(G) = (A^+(G), A^-(G), A^0(G))$, where

\begin{align*}
A^+(G) &= \{uxv \in A(G) | \gamma_R(G^u_xv) > \gamma_R(G)\} \\
A^-(G) &= \{uxv \in A(G) | \gamma_R(G^u_xv) < \gamma_R(G)\} \\
A^0(G) &= \{uxv \in A(G) | \gamma_R(G^u_xv) = \gamma_R(G)\}.
\end{align*}

Before going further, we need to show that an edge lift on any induced path $P_3$ in $A(G)$ can decrease the Roman domination number by at most one and increase it by at most two.

**Theorem 1** For every graph $G$ and every path $uxv \in A(G)$,

\[ \gamma_R(G) - 1 \leq \gamma_R(G^u_xv) \leq \gamma_R(G) + 2. \]

![Graph G](image)

**Figure 1** – Graph $G$.

It can be noted that the upper and lower bounds of Theorem 1 are sharp as can be seen by the graph $G$ in Figure 1 where $\gamma_R(G) = 5$, $\gamma_R(G^{ab}_c) = 7$ and $\gamma_R(G^{cd}_e) = 4$.

2. $\gamma_{RL}^+$-CRITICAL GRAPHS

We begin by giving a necessary condition for $\gamma_{RL}^+$-critical graphs.

**Theorem 2** If $G$ is a connected $\gamma_{RL}^+$-critical graph, then for every $\gamma_R(G)$-function $f = (V_0, V_1, V_2)$ we have:

1) $V_1 = \emptyset$.
2) $V_2$ is an independent set.
3) For all $x \in V_2$, $|epn(x, V_2)| \geq 2$.
4) For all $x \in V_0$, $|N(x) \cap V_2| \leq 2$.

We note that the converse of Theorem 2 is not true as can be seen by the graph $H$ in Figure 2 which satisfies the four conditions of Theorem 2 but $\gamma_R(H^{ac}_b) = 4$ and $\gamma_R(H) = 4$. The next corollary is immediate from Theorem 2 (1).

**Corollary 3** If $G$ is a connected $\gamma_{RL}^+$-critical graph, then $\gamma_R(G)$ is even.

In [2], Cockayne et al. showed that for every graph $G$, $\gamma_R(G) \leq 2\gamma(G)$, and called the graphs attaining equality *Roman graphs*. Moreover, they gave a necessary and sufficient condition for Roman graphs.

**Proposition 4** ([2]) A graph $G$ is Roman if and only if it has a $\gamma_R$-function $f = (V_0, V_1, V_2)$ with $V_1 = \emptyset$. ICMA2021-2
Our next result shows that $\gamma_{RL}^+(G)$-critical graphs $G$ are Roman, that is $\gamma_R(G) = 2\gamma(G)$.

**Proposition 5** If $G$ is $\gamma_{RL}^+(G)$-critical, then $G$ is a Roman graph.

Our next aim is to give a constructive characterization of $\gamma_{RL}^+$-critical trees. We first show that $\gamma_{RL}^+$-critical trees have unique minimum Roman dominating functions. For this purpose we recall the following result due to Gunther et al. [8].

**Theorem 6 (8)** A tree $T$ of order at least three has a unique minimum dominating set $D$ if and only if every vertex in $D$ has at least two private neighbors other than itself.

**Proposition 7** If $T$ is a $\gamma_{RL}^+$-critical tree, then $T$ has a unique $\gamma_R(T)$-function.

We note that the converse of Proposition 7 is not true. To see, consider the double star $T = S_{p,q}$, with $p \geq q \geq 3$, where $T$ has a unique $\gamma_R(T)$-function but $T$ is not $\gamma_{RL}^+$-critical. Let $\mathcal{F}$ be the family of trees $T = T_k$ that can be obtained as follows. Let $T_1$ be a star $K_{1,t}$ (t ≥ 2) centered at $x$ and let $f_1 = (\{y, 0, \{x\}\})$ be an RDF of $T_1$. If $k > 1$, then $T_{k+1}$ can be obtained recursively from $T_k$ by one of the following two operations. For any tree $T_k$ of $\mathcal{F}$, we let $f_i = (V_0^i, V_1^i, V_2^i)$.

- Operation $O_1$: Add a star $K_{1,t}$ (t ≥ 3) centered at $x$ by joining a leaf $z$ of the star to a vertex $h$ of $V_2^i$ such that $epn(h, V_2^i) = N_T(x) \cap V_0^i$. Let $f_{k+1}(y) = f_k(y)$ for every $y \in V(T_k)$, $f_{k+1}(s) = 2$ and $f_{k+1}(u) = 0$ for each neighbor $u$ of $s$.
- Operation $O_2$: Add a star $K_{1,t}$ (t ≥ 2) centered at $s$ by joining a leaf $z$ of the star to a leaf $d$ of $T_k$. Let $f_{k+1}(y) = f_k(y)$ for every $y \in V(T_k)$, $f_{k+1}(s) = 2$ and $f_{k+1}(u) = 0$ for each neighbor $u$ of $s$.

Using both operation we could prove:

**Theorem 8** A tree $T$ of order at least 3 is $\gamma_{RL}^+$-critical if and only if $T \in \mathcal{F}$.

### 3. $\gamma_{RL}^+$-CRITICAL GRAPHS

We begin by giving a necessary and sufficient condition for induced paths $P_3$ of a graph $G$ to be in $A^−(G)$.

**Theorem 9** Let $G$ be a graph and let u$x$v be an induced path $P_3$ in $G$. Then $u$x$v \in A^−(G)$ if and only if there is a $\gamma_R(G)$-function $g = (V_0^g, V_1^g, V_2^g)$ such that:

1. One of $v$ and $u$ belongs to $V_1^g$, say $v$.
2. $u \in V_2^g$. Moreover, if $N(u) \cap V_2^g \neq \emptyset$, then $\vert epn(u, V_2^g)\vert \geq 2$.
3. $x \in V_0^g$ and $\vert N(x) \cap V_2^g\vert \geq 2$.

As an immediate consequence to Theorem 9, we obtain the following corollary.

**Corollary 10** If $G$ is a $\gamma_{RL}^+$-critical graph, then for every induced path $u$x$v$ in $G$, we have:
1) \( \text{deg}_G(x) \geq 3 \).
2) \( \max\{\text{deg}_G(u), \text{deg}_G(v)\} \geq 2 \).

In the sequel, we shall show that no graph with at most one cycle is a \( \gamma_{RL} \)-critical. Let us start with the class of trees

**Theorem 11** No tree on at least three vertices is \( \gamma_{RL} \)-critical.

Recall that the corona of the graph \( G \), denoted by \( G \circ K_1 \), is the graph formed from \( G \) by adding a new vertex \( v' \) and the pendant edge \( vv' \) for each \( v \in V(G) \).

**Theorem 12** No unicycle graph is \( \gamma_{RL} \)-critical.

According to Theorems 11 and 12, one wonders if there are \( \gamma_{RL} \)-critical graphs. The answer is yes as it can be seen by the graph \( G^* \) illustrated in Figure 3. Furthermore, we conjecture that \( G^* \) is the only cactus graph which is \( \gamma_{RL} \)-critical.

![Figure 3 – Graph \( G^* \)](image)

4. REFERENCES


