THE EFFECT OF EDGE LIFTING ON ROMAN DOMINATION IN GRAPHS

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ABSTRACT

Let G be a graph, and let uxv be an induced path centered at x. An edge lift defined on uxv is the action of removing edges ux and vx while adding the edge uv to the edge set of G. In this paper, we initiate the study of the effects of edge lifting on the Roman domination number of a graph, where various properties are established. A characterization of all trees for which every edge lift increases the Roman domination number is provided. Moreover, we characterize the edge lift of a graph decreasing the Roman domination number, and we show that there are no graphs with at most one cycle for which every possible edge lift can have this property.

1. INTRODUCTION

We consider simple graphs G with vertex set $V = V(G)$ and edge set $E = E(G)$. The *open neighborhood* of a vertex $v \in V$ is the set $N_G(v) = N(v) = \{u \in V \mid uv \in E\}$, the *closed neighborhood* of v is the set $N_G[v] = N[v] = N_G(v) \cup \{v\}$, and the *degree* of v is $\deg_G(v) = |N(v)|$.A vertex of degree one is called a*leaf* and its neighbor a *support vertex.* If v is a support vertex, then L_v will denote the set of the leaves attached at v. An *S-external private neighbor* of a vertex $v \in S$ is a vertex $u \in V \setminus S$ which is adjacent to v but to no other vertex of S. The set of all S-external private neighbors of $v \in S$ is called the *S-external private neighbor set* of v and is denoted $\mathrm{epn}(v,S)$. Let uxv be an induced path in a graph G. We say that ux and vx are *lifted off* x and call the operation of removing ux and vx and adding uv to $E(G)$ an *edge lift*. Moreover, the graph formed from G by an edge lift on uxv is denoted as G_x^{uv}, where $V(G_x^{uv}) = V(G)$ and $E(G_x^{uv}) = \{uv\} \cup E(G) \setminus \{ux, vx\}$. The process of *edge lifting* (also called *edge splitting*) was introduced by Lovász in [6, 7] to study edge connectivity in graphs, and has been extended in 2011 to domination in graphs (see [3, 4]) and in 2013 to total domination in graphs (see [5]) to study its effects on the corresponding domination parameters. Our aim is to study the effect of edge lifting with respect to Roman domination. A function $f : V \to \{0,1,2\}$ is a *Roman dominating function*, abbreviated RDF, on a graph G if for every vertex $v \in V$ with $f(v) = 0$, there exists a neighbor $u \in N(v)$ with $f(u) = 2$. The *weight* of an RDF f is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of an RDF on G is called the *Roman domination number* of G and is denoted by $\gamma_R(G)$. An RDF on G with weight $\gamma_R(G)$ is called a $\gamma_R(G)$-*function.* Notice that an RDF f on G can be denoted by (V_0,V_1,V_2) (or (V_0^f,V_1^f,V_2^f) to refer to f), where $V_j = \{v \in V \mid f(v) = j\}$ for $j \in \{0,1,2\}$. The Roman domination number was introduced by Cockayne et al. [2] in 2004 and was inspired by the work of ReVelle and Rosing [9] and Stewart [10]. For more information on Roman domination, we refer the reader to the book chapter [1]. All graphs considered are connected and contain at least one induced path on three vertices. For every graph G we define the following weak partition (in which a subset may be empty) $A(G) = (A^+(G), A^-(G), A^0(G))$, where

$$A^+(G) = \{uxv \in A(G) | \gamma_R(G_x^u v) > \gamma_R(G)\}$$
$$A^-(G) = \{uxv \in A(G) | \gamma_R(G_x^u v) < \gamma_R(G)\}$$
$$A^0(G) = \{uxv \in A(G) | \gamma_R(G_x^u v) = \gamma_R(G)\}.$$

Before going further, we need to show that an edge lift on any induced path P_3 in $A(G)$ can decrease the Roman domination number by at most one and increase it by at most two.

Theorem 1 *For every graph G and every path $uxv \in A(G)$,*

$$\gamma_R(G) - 1 \leq \gamma_R(G_x^u v) \leq \gamma_R(G) + 2.$$

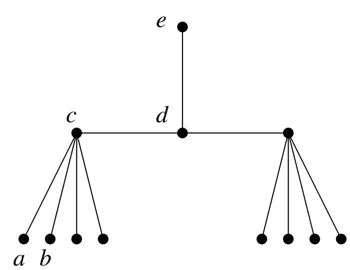

FIGURE 1 – Graph G.

It can be noted that the upper and lower bounds of Theorem 1 are sharp as can be seen by the graph G in Figure 1, where $\gamma_R(G) = 5$, $\gamma_R(G_c^{ab}) = 7$ and $\gamma_R(G_d^{ec}) = 4$.

2. γ_{RL}^+–CRITICAL GRAPHS

We begin by giving a necessary condition for γ_{RL}^+-critical graphs.

Theorem 2 *If G is a connected γ_{RL}^+-critical graph, then for every $\gamma_R(G)$-function $f = (V_0, V_1, V_2)$ we have :*

1) *$V_1 = \emptyset$.*
2) *V_2 is an independent set.*
3) *For all $x \in V_2$, $|\text{epn}(x, V_2)| \geq 2$.*
4) *For all $x \in V_0$, $|N(x) \cap V_2| \leq 2$.*

We note that the converse of Theorem 2 is not true as can be seen by the graph H in Figure 2 which satisfies the four conditions of Theorem 2 but $\gamma_R(H_b^{ac}) = \gamma_R(H) = 4$. The next corollary is immediate from Theorem 2-(1).

Corollary 3 *If G is a connected γ_{RL}^+-critical graph, then $\gamma_R(G)$ is even.*

In [2], Cockayne et al. showed that for every graph G, $\gamma_R(G) \leq 2\gamma(G)$, and called the graphs attaining equality *Roman graphs*. Moreover, they gave a necessary and sufficient condition for Roman graphs.

Proposition 4 ([2]) *A graph G is Roman if and only if it has a γ_R-function $f = (V_0, V_1, V_2)$ with $V_1 = \emptyset$.*

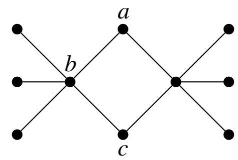

FIGURE 2 – Graph H.

Our next result shows that $\gamma_{RL}^{+}(G)$-critical graphs G are Roman, that is $\gamma_R(G) = 2\gamma(G)$.

Proposition 5 *If G is $\gamma_{RL}^{+}(G)$-critical, then G is a Roman graph.*

Our next aim is to give a constructive characterization of γ_{RL}^{+}-critical trees. We first show that γ_{RL}^{+}-critical trees have unique minimum Roman dominating functions. For this purpose we recall the following result due to Gunther et al. [8].

Theorem 6 ([8]) *A tree T of order at least three has a unique minimum dominating set D if and only if every vertex in D has at least two private neighbors other than itself.*

Proposition 7 *If T is a γ_{RL}^{+}-critical tree, then T has a unique $\gamma_R(T)$-function.*

We note that the converse of Proposition 7 is not true. To see, consider the double star $T = S_{p,q}$, with $p \geq q \geq 3$, where T has a unique $\gamma_R(T)$-function but T is not γ_{RL}^{+}-critical. Let $\mathscr{F}$ be the

family of trees $T = T_k$ that can be obtained as follows. Let T_1 be a star $K_{1,t}$ $(t \geq 2)$ centered at x and let $f_1 = (L_x, \emptyset, \{x\})$ be an RDF of T_1. If $k > 1$, then T_{i+1} can be obtained recursively from T_i by one of the following two operations. For any tree T_i of $\mathscr{F}$, we let $f_i = (V_0^i, V_1^i, V_2^i)$.

— Operation $\mathscr{O}_1$: Add a star $K_{1,t}$ $(t \geq 3)$ centered at s by joining a leaf z of the star to a vertex h of V_2^i such that $\text{epn}(h, V_2^i) = N_{T_i}(x) \cap V_0^i$. Let $f_{i+1}(y) = f_i(y)$ for every $y \in V(T_i)$, $f_{i+1}(s) = 2$ and $f_{i+1}(u) = 0$ for each neighbor u of s.

— Operation $\mathscr{O}_2$: Add a star $K_{1,t}$ $(t \geq 2)$ centered at s by joining a leaf z of the star to a leaf d of T_i . Let $f_{i+1}(y) = f_i(y)$ for every $y \in V(T_i)$, $f_{i+1}(s) = 2$ and $f_{i+1}(u) = 0$ for each neighbor u of s.

Using both operation we could prove :

Theorem 8 *A tree T of order at least 3 is γ_{RL}^{+}-critical if and only if $T \in \mathscr{F}$.*

3. γ_{RL}^{-}-CRITICAL GRAPHS

We begin by giving a necessary and sufficient condition for induced paths P_3 of a graph G to be in $A^{-}(G)$.

Theorem 9 *Let G be a graph and let uxv be an induced path P_3 in G. Then $uxv \in A^{-}(G)$ if and only if there is a $\gamma_R(G)$-function $g = (V_0^g, V_1^g, V_2^g)$ such that :*

1) One of v and u belongs to V_1^g, say v.
2) $u \in V_2^g$. Moreover, if $N(u) \cap V_2^g \neq \emptyset$, then $|\text{epn}(u, V_2^g)| \geq 2$.
3) $x \in V_0^g$ and $|N(x) \cap V_2^g| \geq 2$.

As an immediate consequence to Theorem 9 we obtain the following corollary.

Corollary 10 *If G is a γ_{RL}^{-}-critical graph, then for every induced path uxv in G, we have :*

1) $\deg_G(x) \geq 3$.
2) $\max\{\deg_G(u), \deg_G(v)\} \geq 2$.

In the sequel, we shall show that no graph with at most one cycle is a γ_{RL}^{-}-critical. Let us start with the class of trees

Theorem 11 *No tree on at least three vertices is γ_{RL}^{-}-critical.*

Recall that the *corona* of the graph G, denoted by $G \circ K_1$, is the graph formed from G by adding a new vertex v' and the pendant edge vv' for each $v \in V(G)$.

Theorem 12 *No unicycle graph is γ_{RL}^{-}-critical.*

According to Theorems 11 and 12, one wonders if there are γ_{RL}^{-}-critical graphs. The answer is yes as it can be seen by the graph G^* illustrated in Figure 3. Furthermore, we conjecture that G^* is the only cactus graph which is γ_{RL}^{-}-critical.

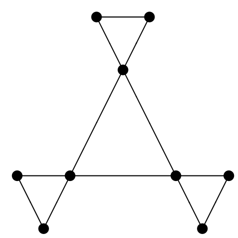

FIGURE 3 – Graph G^*

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