SOME EXISTENCE RESULTS TO POSITIVE SOLUTIONS FOR P-LAPLACIAN BOUNDARY VALUE PROBLEMS OF FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In this manuscript, we deal with a study of the existence and the multiplicity of ρ_1-concave positive solutions for a boundary value of two-sided fractional differential equations involving generalized-Caputo fractional derivatives. An application of a functional analysis tools, more specifically, we using some fixed point theorems and under some additional assumptions, some of important results have been proven and we obtain the existence of at least one solution.

Key Words : Fractional differential equations ; Caputo-Katugampola (CK) fractional type ; boundary value problem ; fixed point theorems ; Positive solutions.

1. INTRODUCTION

Lately, many researches on fractional differential problems have been dealt with by many researchers. Especial, fractional p-Laplacian has been used in modeling different problems, for exampale in science, engineering, biology, [6, 8, 9, 10, 11, 19] etc.

In this words, we consider the following fractional boundary value problem (FBVP) :

$$\begin{cases} {}^{\rho_2;C}\mathscr{D}^{\sigma}_{T^-}\left(\phi_p\left({}^{\rho_1;C}\mathscr{D}^{\alpha}_{a^+}u\right)\right)(t)+cf(u(t))=0, & a<t<T,\\ u(a)+{}^{\rho_1;C}\mathscr{D}^{\alpha}_{a^+}u(a)=0,\ \delta^2_{\rho_1}u(a)=0,\ \delta^1_{\rho_1}u(T)=\mu\delta^1_{\rho_1}u(\eta)+\lambda, & \\ {}^{\rho_1;C}\mathscr{D}^{\alpha}_{a^+}u(T)=-\delta^1_{\rho_2}\left[\phi_p\left({}^{\rho_1;C}\mathscr{D}^{\alpha}_{a^+}u\right)\right](a)=\delta^2_{\rho_2}\left[\phi_p\left({}^{\rho_1;C}\mathscr{D}^{\alpha}_{a^+}u\right)\right](T)=0, & \end{cases} \tag{1}$$

where

— ${}^{\rho_1;C}\mathscr{D}^{\sigma}_{a^+}, {}^{\rho_2;C}\mathscr{D}^{\alpha}_{T^-}$ are the left and right-sided Caputo-Katugampola fractional derivatives with $\rho_1,\rho_2\in\mathbb{R}-\{1\}$ and $2<\sigma,\alpha\leqslant 3$,

— ϕ_p is the p-Laplacian operator, i.e., $\phi_p(s)=|s|^{p-2}s, p>1$, $\delta^k_{\rho}=\left(t^{1-\rho}\dfrac{\mathrm{d}}{\mathrm{d}t}\right)^k$,

— f is continuous and positive.

— $\eta\in(a,T)$, $0\leq\mu<1$, $c\geq 0$ and $\lambda\geq 0$.

In [15], Chuanzhi Bai used the Guo-Krasnoselskii fixed point theorem and the Banach contraction mapping principle to prove the existence and uniqueness of positive solutions for the FBVP :

$$\begin{cases} \left(\phi_p\left(D^{\alpha}_{0^+}u\right)\right)'(t)+f(t,u(t))=0, \qquad 0<t<1,\\ u(0)=D^{\alpha}_{0^+}u(0)=0, \quad {}^{c}D^{\beta}_{0^+}u(0)={}^{c}D^{\beta}_{0^+}u(1)=0, \end{cases} \tag{2}$$

where $0<\beta\leq 1, 2<\alpha<2+\beta, D_{0^+}^{\alpha}$ and ${}^cD_{0^+}^{\beta}$ are the Riemann-Liouville and Caputo fractional derivatives of orders α,β, respectively, $p>1$, and $f:[a,b]\times\mathbb{R}\to\mathbb{R}$ is a continuous function.

Chai [1], obtained some results for the existence of at least one nonnegative solution and two positive solutions by using a fixed point theorem on a cone for the nonlinear FBVP

$$\begin{cases} D_{0+}^{\beta}\left(\phi_p\left(D_{0+}^{\alpha}u\right)\right)(t)+f(t,u(t))=0, \quad 0<t<1,\\ u(0)=0, u(1)+\sigma D_{0+}^{\gamma}u(1)=0, \quad D_{0+}^{\alpha}u(0)=0, \end{cases} \tag{3}$$

where $1<\alpha\leq 2$, $0<\beta,\gamma\leq 1$, $0\leq\alpha-\gamma-1$, and $\sigma>0$.

Using the fixed point index theory, Su et al. [13] studied the existence of positive solution for a nonlinear four-point singular FBVP

$$\begin{cases} \left(\phi_p\left(u'\right)\right)'(t)+a(t)f(u(t))=0, \quad 0<t<1,\\ \alpha\phi_p(u(0))-\beta\phi_p\left(u'(\xi)\right)=0, \qquad \gamma\phi_p(u(1))+\delta\phi_p\left(u'(\eta)\right)=0, \end{cases} \tag{4}$$

where $\alpha,\gamma>0,\beta,\delta\geq 0,\xi,\eta\in(0,1)$, and $\xi<\eta$.

Su [12] applied the fixed-point index theory to study the existence of positive solutions for the nonlinear third-order two-Point singular boundary value problem

$$\begin{cases} \left(\phi_p\left(u^{(n-1)}\right)\right)'(t)+a(t)f(u(t))=0, \quad 0<t<1,\\ u(0)=u'(0)=\cdots=u^{(n-3)}(0)=u^{(n-1)}(0)=0, \quad u(1)=\sum_{i=1}^{m-2}\alpha_i u\left(\eta_i\right), \end{cases} \tag{5}$$

where $0<\eta_1<\eta_2<\cdots<\eta_{m-2}<1, \alpha_i>0$ with $\sum_{i=1}^{m-2}\alpha_i\eta_i^{n-2}<1$.

Using the coincidence degree theory, Tang et al. [14] gave a new result on the existence of positive solutions to the FBVP

$$\begin{cases} {}^cD_{0^+}^{\alpha}\left(\phi_p{}^cD_{0^+}^{\beta}u\right)(t)=f\left(t,u(t),{}^cD_{0^+}^{\beta}u(t)\right),\\ u(0)=0, \quad {}^cD_{0^+}^{\beta}u(0)={}^cD_{0^+}^{\beta}u(1), \end{cases} \tag{6}$$

where $0<\alpha,\beta\leq 1, 1<\alpha+\beta\leq 2$.

Torres [16], studied the existence and multiplicity for a mixed-order three-point boundary value problem of fractional differential equation evolving Caputo's differential operator and the boundary conditions with integer order derivatives

$$\begin{cases} \left(\varphi_p\left({}^cD_{0^+}^{\alpha}u\right)\right)'(t)+a(t)f(t,u(t))=0, \quad 0<t<1,\\ {}^cD_{0^+}^{\alpha}u(0)=u(0)=u''(0)=0, \quad u'(1)=\gamma u'(\eta), \end{cases} \tag{7}$$

where $\eta,\gamma\in(0,1),\alpha\in(2,3]$.

Base on the coincidence degree theory, Chen et al. [2] gave new results about the problem

$$\begin{cases} {}^cD_{0^+}^{\beta}\phi_p\left({}^cD_{0^+}^{\alpha}x\right)(t)=f\left(t,x(t),{}^cD_{0^+}^{\alpha}x(t)\right), t\in[0,1],\\ {}^cD_{0^+}^{\alpha}x(0)={}^cD_{0^+}^{\alpha}x(1)=0, \end{cases} \tag{8}$$

where $0<\alpha,\beta\leq 1, 1<\alpha+\beta\leq 2$.

It can be seen that our work presented in this words has some following features which are different from those [2, 16, 14, 12, 13, 1, 15], in addition to this, the appropriate manipulations of the controls ρ_1, ρ_2, α, σ, c, λ, μ and η, allow us to see the impact and the extent of our study. Add to this there is no known research paper that delves deeper than us into extracting some of the properties of the Greens functions which used to study the existence of solutions, special when the non-integer order of the fractional derivatives is huge.

In this words, we obtain some sufficient conditions ensuring the existence of at least one positive solutions for the BVP (1). The rest of the paper is organized as follows. Section 2 presents some basic definitions, lemmas, and preliminary results. In Section 3, we present some important lemmas. In Section 4, we derive some conditions on the parameter λ to obtain the existence of at least one positive solution. Finally, we give some illustrative examples in Section 5.

2. PRELIMINARIES AND BACKGROUND MATERIALS

Notations

In addition to the notations introduced with the problem (1), let $J = [a,T] \subset (0,\infty)$, and $\rho > 0$.

1. $C(J)$ denotes the Banach space of continuous functions h on J endowed with the norm
$$\|h\|_C = \max_{x\in J} |h(x)|.$$
2. $AC(J)$ and $C^n(J)$ denote the spaces of absolutely continuous and n times continuously differentiable functions on J respectively.
3. $L^p(a,T)$ denotes the space of Lebesgue integrable functions on (a,T).
4. $C^n_\rho(J)$ is the Banach space of n continuously differentiable functions on J, with respect to δ_ρ :
$$C^n_\rho(J) = \left\{ h \in C(J) : \ \delta^k_\rho h \in C(J), k = 0,1,...,n \right\},$$
endowed with the norm
$$\|h\|_{C^n_\rho} = \sum_{k=0}^{n} \|\delta^k_\rho h\|_C.$$
5. $[\alpha]$ is the largest integer less than or equal to α. Throughout the words, we use $n = [\alpha]+1$.
6. $C^+(J) = \{y \in C(J),\ y(t) \geq 0 \ \ \forall t \in J\}$.

2.1. Fractional calculus

We present here basic definitions and lemmas from fractional calculus theory (See [6, 8, 9, 10, 11, 5] and references therein).

Definition 1 (Function space) *For $c \in \mathbb{R}$, consider the Banach space*

$$\mathscr{X}^p_c(a,T) = \left\{ h : J \to \mathbb{R} : \ \|h\|_{\mathscr{X}^p_c} := \left(\int_a^T |\, t^c h(t)\,|^p \, \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p}} < +\infty \right\}.$$

Now, we recall the Katugampola (K) and Caputo-Katugampola (CK) fractional integrals and derivatives [5].

Definition 2 *For a function $u \in \mathscr{X}^p_c(a,T)$, the Katugampola left-sided (${}^\rho\mathscr{I}^\alpha_{a^+}$) and right-sided (${}^\rho\mathscr{I}^\alpha_{T^-}$) fractional integrals of order $\alpha > 0$ are defined by*

$${}^\rho\mathscr{I}^\alpha_{a^+} u(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} u(s) ds, \quad t > a,$$

$${}^\rho\mathscr{I}^\alpha_{T^-} u(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_t^T (s^\rho - t^\rho)^{\alpha-1} s^{\rho-1} u(s) ds, \quad t < T.$$

Remark 1 *For a function $u \in \mathscr{X}^p_c(a,T)$, the Katugampola left-sided (${}^\rho\mathscr{I}^\alpha_{a^+}$) fractional integrals of order $\alpha > 0$ can be geven by the generalized convolution of u and h*

$${}^\rho\mathscr{I}^\alpha_{a^+} u(t) = \left(h \star_g u\right)(t) = \int_a^t h(s) u(g^{-1}\left(g(t) + g(a) - g(s)\right) g'(s) ds, \quad t > a,$$

where $h(t) = \frac{1}{\Gamma(\alpha)} (g(t) - g(a))^{\alpha-1}$ and $g(t) = \frac{t^\rho}{\rho}$. For more detailes see [20]. This generalized convolution of two functions is commutative.

Next, we present some properties for left-sided integrals. But, the same properties are also true for the right-sided ones.

Lemma 1 *Let $c \in \mathbb{R}$, $\alpha, \beta, \rho > 0$, and $1 \leq p \leq \infty$. Then, for all $u \in \mathscr{X}_c^p(a,T)$, we have the following.*

— ${}^{\rho}\mathscr{I}_{a^+}^{\alpha} : \mathscr{X}_c^p(a,T) \to \mathscr{X}_c^p(a,T)$.

— ${}^{\rho}\mathscr{I}_{a^+}^{\alpha}$ *is linear.*

— ${}^{\rho}\mathscr{I}_{a^+}^{\alpha} \circ {}^{\rho}\mathscr{I}_{a^+}^{\beta} = {}^{\rho}\mathscr{I}_{a^+}^{\alpha+\beta}$.

Definition 3 *For a function $u \in C_{\delta}^n([a,T])$ (or $\in AC_{\delta}^n([a,T])$), the Caputo-Katugampola fractional derivatives are defined by :*

$$ {}^{\rho_1;C}\mathscr{D}_{a^+}^{\alpha} u(t) = {}^{\rho_1}\mathscr{I}_{a^+}^{n-\alpha} \delta_{\rho_1}^n u(t).$$

$$ {}^{\rho_1;C}\mathscr{D}_{T^-}^{\alpha} u(t) = (-1)^n \, {}^{\rho_1}\mathscr{I}_{T^-}^{n-\alpha} \delta_{\rho_1}^n u(t).$$

Lemma 2 *Let $\beta > \alpha > 0$, $h \in \mathscr{M}_c^p(a,T)$, $u \in AC_{\delta}^n([a,T])$ or $C_{\delta}^n([a,T])$. Then we have*

$$ {}^{\rho;C}\mathscr{D}_{a^+}^{\alpha} \, {}^{\rho}\mathscr{I}_{a^+}^{\beta} h(t) = {}^{\rho}\mathscr{I}_{a^+}^{\beta-\alpha} h(t), \tag{9}$$

$$ {}^{\rho}\mathscr{I}_{a^+}^{\alpha} \left({}^{\rho;C}\mathscr{D}_{a^+}^{\alpha} u \right)(t) = u(t) - \sum_{k=0}^{n-1} C_k \left(\frac{t^{\rho} - a^{\rho}}{\rho} \right)^k, \tag{10}$$

$$ {}^{\rho}\mathscr{I}_{T^-}^{\alpha} \left({}^{\rho;C}\mathscr{D}_{T^-}^{\alpha} u \right)(t) = u(t) - \sum_{k=0}^{n-1} D_k \left(\frac{T^{\rho} - t^{\rho}}{\rho} \right)^k, \tag{11}$$

for some real constants C_k and D_k.

Lemma 3 *If ${}^{\rho;C}\mathscr{D}_{a^+}^{\alpha} u \in C(J)$, then $u \in C_{\rho}^{n-1}(J)$.*

Lemma 4 *If $u \in C^1(J)$, then*

$$ \delta_{\rho}^1 \left({}^{\rho}\mathscr{I}_{a^+}^{\alpha} u \right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{s^{\rho} - a^{\rho}}{\rho} \right)^{\alpha-1} \frac{s^{\rho-1}}{\left(t^{\rho} + a^{\rho} - s^{\rho}\right)^{\frac{\rho-1}{\rho}}} u'\left((t^{\rho} + a^{\rho} - s^{\rho})^{\frac{1}{\rho}} \right) \mathrm{d}s \in C(J)$$

.

2.2. Fixed point theorems

Let $\mathbb{E}$ be a real Banach function space, endowed with the infinity norm.

Definition 4 *A nonempty closed convex set $K \subset \mathbb{E}$ is called cone if the following properties are satisfied.*

1. $\forall u \in K, \forall \lambda > 0 : \ \lambda u \in K$.
2. $\forall u \in K : -u \in K \implies u = 0$.

Definition 5 *A continuous operator is called completely continuous if it maps bounded sets into precompact sets.*

Let $K \subset \mathbb{E}$ be a cone, $r > 0$, $\mathscr{P}_r = \{u \in K : \|u\| < r\}$, and $\mathbf{i}$ is the fixed point index function.

Theorem 5 ([4],[3]) *Let $\mathscr{L} : K \cap \overline{\mathscr{P}}_r \to K$ be a completely continuous operator such that $\mathscr{L}u \neq u$ for all $u \in \partial \mathscr{P}_r$.*

1. *If $\|\mathscr{L}u\| \leqslant \|u\|$ for all $u \in \partial \mathscr{P}_r$, then $\mathbf{i}(\mathscr{L}, \Omega_r, K) = 1$.*
2. *If $\|\mathscr{L}u\| \geqslant \|u\|$ for all $u \in \partial \mathscr{P}_r$, then $\mathbf{i}(\mathscr{L}, \Omega_r, K) = 0$.*

Theorem 6 (Guo-Krasnoselskii [6]) *Assume that $\mathscr{P}_1$ and $\mathscr{P}_2$ are open subsets of $\mathbb{E}$ with $0 \in \mathscr{P}_1$ and $\overline{\mathscr{P}_1} \subset \mathscr{P}_2$. Let $\mathscr{L} : K \cap (\overline{\mathscr{P}_2} \setminus \mathscr{P}_1) \to K$ be completely continuous operator. Consider*
(D_1) $\|\mathscr{L}u\| \leq \|u\|, \forall u \in K \cap \partial \mathscr{P}_1$ and $\|\mathscr{L}u\| \geq \|u\|, \forall u \in K \cap \partial \mathscr{P}_2$,
(D_2) $\|\mathscr{L}u\| \leq \|u\|, \forall u \in K \cap \partial \mathscr{P}_2$ and $\|\mathscr{L}u\| \geq \|u\|, \forall u \in K \cap \partial \mathscr{P}_1$.
If (D_1) or (D_2) holds, then $\mathscr{L}$ has a fixed point in $K \cap (\overline{\mathscr{P}_2} \setminus \mathscr{P}_1)$.

3. MAJOR RESULTS

We present some important lemmas which play a key role to prove the major results.

3.1. Lemmas

Consider the linear generalized boundary-value problem associated to (1)

$$\begin{cases} {}^{\rho_1;C}\mathscr{D}^{\alpha}_{a^+}u(t) + y(t) = 0, \quad a < t < T, \\ u(a) + {}^{\rho_1;C}\mathscr{D}^{\alpha}_{a^+}u(a) = 0, \delta^2_{\rho_1}u(a) = 0, \delta^1_{\rho_1}u(T) - \mu\delta^1_{\rho_1}u(\eta) = \lambda. \end{cases} \tag{12}$$

Lemma 7 *For $y \in C(J)$, the integral solution of* (12) *is given by*

$$u(t) = \int_a^T G(t,s)y(s)\mathrm{d}s + \left(\frac{t^{\rho_1} - a^{\rho_1}}{\rho_1(1-\mu)}\right)\mu\int_a^T H(\eta,s)y(s)\mathrm{d}s + \left(\frac{t^{\rho_1} - a^{\rho_1}}{\rho_1(1-\mu)}\right)\lambda + y(a).$$

where

$$G(t,s) = \begin{cases} \frac{1}{\Gamma(\alpha-1)}\left(\frac{t^{\rho_1}-a^{\rho_1}}{\rho_1}\right)\left(\frac{T^{\rho_1}-s^{\rho_1}}{\rho_1}\right)^{\alpha-2}s^{\rho_1-1} - \frac{1}{\Gamma(\alpha)}\left(\frac{t^{\rho_1}-s^{\rho_1}}{\rho_1}\right)^{\alpha-1}s^{\rho_1-1}, & a \leq s \leq t \leq T, \\ \frac{1}{\Gamma(\alpha-1)}\left(\frac{t^{\rho_1}-a^{\rho_1}}{\rho_1}\right)\left(\frac{T^{\rho_1}-s^{\rho_1}}{\rho_1}\right)^{\alpha-2}s^{\rho_1-1}, & a \leq t \leq s \leq T, \end{cases} \tag{13}$$

and

$$H(\eta,s) = \begin{cases} \frac{1}{\Gamma(\alpha-1)}\left(\frac{T^{\rho_1}-s^{\rho_1}}{\rho_1}\right)^{\alpha-2}s^{\rho_1-1} - \frac{1}{\Gamma(\alpha-1)}\left(\frac{\eta^{\rho_1}-s^{\rho_1}}{\rho_1}\right)^{\alpha-2}s^{\rho_1-1}, & a \leq s \leq \eta \leq T, \\ \frac{1}{\Gamma(\alpha-1)}\left(\frac{T^{\rho_1}-s^{\rho_1}}{\rho}\right)^{\alpha-2}s^{\rho_1-1}, & a \leq \eta \leq s \leq T. \end{cases} \tag{14}$$

Proof. By applying (10) and from the boundary conditions of (12) we get the desired result. The converse follows by direct computation. The proof is completed.

■

Now, consider the generalized boundary-value problem associated to (1)

$$\begin{cases} {}^{\rho_2;C}\mathscr{D}^{\sigma}_{T^-}\left(\phi_p\left({}^{\rho_1;C}\mathscr{D}^{\alpha}_{a^+}u(t)\right)\right) = g(t), \quad a < t < T, \\ u(a) + {}^{\rho_1;C}\mathscr{D}^{\alpha}_{a^+}u(a) = 0, \delta^2_{\rho_1}u(a) = 0, \quad \delta^1_{\rho_1}u(T) - \mu\delta^1_{\rho_1}u(\eta) = \lambda, \\ {}^{\rho_1;C}\mathscr{D}^{\alpha}_{a^+}u(T) = 0, -\delta^1_{\rho_2}\left[\phi_p\left({}^{\rho_1;C}\mathscr{D}^{\alpha}_{a^+}u\right)\right](a) = 0, \delta^2_{\rho_2}\left[\phi_p\left({}^{\rho_1;C}\mathscr{D}^{\alpha}_{a^+}u\right)\right](T) = 0. \end{cases} \tag{15}$$

Lemma 8 *Let $g(t) \in C^+(J)$ and $y(t) = \phi_{\bar{p}}\left(\int_a^T K(t,\tau)g(\tau)\mathrm{d}\tau\right)$, the problem* (15) *is equivalent to the problem* (12) *where*

$$K(t,s) = \begin{cases} \frac{1}{\Gamma(\sigma-1)}\left(\frac{T^{\rho_2}-t^{\rho_2}}{\rho_2}\right)\left(\frac{s^{\rho_2}-a^{\rho_2}}{\rho_2}\right)^{\sigma-2} s^{\rho_2-1} - \frac{1}{\Gamma(\sigma)}\left(\frac{s^{\rho_2}-t^{\rho_2}}{\rho_2}\right)^{\sigma-1} s^{\rho_2-1}, & t \le s, \\ \frac{1}{\Gamma(\sigma-1)}\left(\frac{T^{\rho_2}-t^{\rho_2}}{\rho_2}\right)\left(\frac{s^{\rho_2}-a^{\rho_2}}{\rho_2}\right)^{\sigma-2} s^{\rho_2-1}, & s \le t, \end{cases} \tag{16}$$

Proof. From Lemma 2, the equation (15) is equivalent to the equation

$$\phi_p\left({}^{\rho_1;C}\mathscr{D}^{\alpha}_{a^+}u(t)\right) \quad = -D_0 - D_1\left(\frac{T^{\rho_2}-t^{\rho_2}}{\rho_2}\right) - D_2\left(\frac{T^{\rho_2}-t^{\rho_2}}{\rho_2}\right)^2 + {}^{\rho_2}\mathscr{I}^{\sigma}_{T^-}g(t),$$

for some constants $D_0, D_1, D_2 \in \mathbb{R}$. Using the second boundary condition, we get

$$\begin{aligned}\phi_p\left({}^{\rho_1;C}\mathscr{D}^{\alpha}_{a^+}u(t)\right) &= {}^{\rho_2}\mathscr{I}^{\sigma}_{T^-}g(t) - \left(\frac{T^{\rho_2}-t^{\rho_2}}{\rho_2}\right) \times \frac{1}{\Gamma(\sigma-1)}\int_a^T \left(\frac{\tau^{\rho_2}-a^{\rho_2}}{\rho_2}\right)^{\sigma-2} \tau^{\rho_2-1} g(\tau)\mathrm{d}\tau \\ &= -\int_a^T K(t,\tau)q(\tau)\mathrm{d}\tau.\end{aligned}$$

Consequently,

$${}^{\rho_1;C}\mathscr{D}^{\alpha}_{a^+}u(t) = -\phi_{\bar{p}}\left(\int_a^T K(t,\tau)g(\tau)\mathrm{d}\tau\right),$$

where $\bar{p} = \frac{p}{p-1}$. Thus, the problem (15) can be written as

$$\begin{cases} {}^{\rho_1;C}\mathscr{D}^{\alpha}_{a^+}u(t) + \phi_{\bar{p}}\left(\int_a^T K(t,\tau)q(\tau)\mathrm{d}\tau\right) = 0, \quad t \in (a,T), \\ u(a) + {}^{\rho_1;C}\mathscr{D}^{\alpha}_{a^+}u(a) = 0, \delta^2_{\rho_1}u(a) = 0, \quad \delta^1_{\rho_1}u(T) = \mu\delta^1_{\rho_1}u(\eta) + \lambda, \end{cases} \tag{17}$$

which, according to Lemma 7, has a unique solution of the form (18). ■

Lemma 9 *For $g(t) \in C^+(J)$, the BVP* (15) *has a unique solution*

$$\begin{aligned}u(t) = &\int_a^T G(t,s)\phi_{\bar{p}}\left(\int_a^T K(s,\tau)g(\tau)\mathrm{d}\tau\right)\mathrm{d}s + \left(\frac{t^{\rho_1}-a^{\rho_1}}{\rho_1-\mu\rho_1}\right)\mu\int_a^T H(\eta,s)\phi_{\bar{p}}\left(\int_a^T K(s,\tau)g(\tau)\mathrm{d}\tau\right)\mathrm{d}s \\ &+ \left(\frac{t^{\rho_1}-a^{\rho_1}}{\rho_1-\mu\rho_1}\right)\lambda + \phi_{\bar{p}}\left(\int_a^T K(a,\tau)g(\tau)\mathrm{d}\tau\right),\end{aligned} \tag{18}$$

where $G(t,s)$, $H(t,s)$ and $K(t,s)$ are defined in the previous Lemmas 7.

Lemma 10 *The functions G, H and K (Equ.* (13)*,* (14)*, and* (16)*) satisfy the following.*

(i) *$G(t,s)$, $H(t,s)$ and $K(t,s)$ are continuous on $[a,T]\times[a,T]$.*

(ii) *$\forall (t,s) \in [a,T]\times[a,T]$:*

$$G(t,s) \leqslant \left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha-1}\frac{T^{\rho_1-1}}{\Gamma(\alpha-1)}, \int_a^T G(t,s)\mathrm{d}s = \left(\frac{t^{\rho_1}-a^{\rho_1}}{\Gamma(\alpha)\rho_1}\right)\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha-1} - \frac{1}{\Gamma(\alpha+1)}\left(\frac{t^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha},$$

$$H(n,s) \leqslant \left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha-2}\frac{T^{\rho_1-1}}{\Gamma(\alpha-1)}, \int_a^T H(n,s)\mathrm{d}s = \frac{1}{\Gamma(\alpha)}\left(\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha-1} - \left(\frac{\eta^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha-1}\right),$$

$$K(t,s) \leqslant \left(\frac{T^{\rho_2}-a^{\rho_2}}{\rho_2}\right)^{\sigma-1}\frac{T^{\rho_2-1}}{\Gamma(\sigma-1)}, \int_a^T K(s,\tau)\mathrm{d}\tau = \frac{T^{\rho_2}-s^{\rho_2}}{\rho_2\Gamma(\sigma)}\left(\left(\frac{T^{\rho_2}-a^{\rho_2}}{\rho_2}\right)^{\sigma-1} - \frac{1}{\sigma}\left(\frac{T^{\rho_2}-s^{\rho_2}}{\rho_2}\right)^{\sigma-1}\right).$$

(iii) $\forall (t,s)\in [a,T]^2:\quad G(t,s)\geqslant 0, H(t,s)\geqslant 0, K(t,s)\geqslant 0$;

(iv) *The function* $t\to G(t,s)$ *is increasing and* $t\to K(t,s)$ *is decreasing* $\forall s\in[a,T]$. *In addition, we have*

$$\forall (t,s)\in (a,T)^2:\left(\frac{t^{\rho_1}-a^{\rho_1}}{T^{\rho_1}-a^{\rho_1}}\right)^{\alpha-1}G(T,s)\leqslant G(t,s)\ \text{ and }\ \left(\frac{T^{\rho_2}-t^{\rho_2}}{T^{\rho_2}-a^{\rho_2}}\right)^{\sigma-1}K(a,s)\leqslant K(t,s);$$

(v) $\forall (t,s)\in (a,T)^2:\quad \frac{t^{\rho_1-1}\rho_1}{T^{\rho_1}-a^{\rho_1}}\left[1-\left(\frac{t}{T}\right)^{\rho_1(\alpha-2)}\right]G(T,s)\leqslant G'_t(t,s)\leqslant \frac{\alpha-1}{\alpha-2}\frac{t^{\rho_1-1}\rho_1}{T^{\rho_1}-a^{\rho_1}}G(T,s)$ *and* $\mid K'_t(t,s)\mid\leqslant \frac{t^{\rho_2-1}\rho_2}{T^{\rho_2}-a^{\rho_2}}K(a,s)$.

Proof. Using the definitions of G, H and K, (i) and (ii) are obtained straightforwardly.

— Property (iii). We only consider the case $s\leq t$ as the other case is straightforward. When $s\leq t$, we have

$$\begin{aligned}G(t,s)&\geqslant \frac{1}{\Gamma(\alpha-1)}\left(\frac{t^{\rho_1}-s^{\rho_1}}{\rho_1}\right)\left(\frac{t^{\rho_1}-s^{\rho_1}}{\rho_1}\right)^{\alpha-2}a^{\rho_1-1}-\frac{1}{\Gamma(\alpha)}\left(\frac{t^{\rho_1}-s^{\rho_1}}{\rho_1}\right)^{\alpha-1}a^{\rho_1-1},\\&\geqslant \left(\frac{t^{\rho_1}-s^{\rho_1}}{\rho_1}\right)^{\alpha-1}a^{\rho_1-1}\left[\frac{1}{\Gamma(\alpha-1)}-\frac{1}{\Gamma(\alpha)}\right],\\&\geqslant 0,\qquad\qquad \Big(\text{because }\Gamma(\alpha-1)\leqslant\Gamma(\alpha)\ \text{ for } 2<\alpha\leq 3\Big).\end{aligned}$$

Similarly, we can easily prove that $H(t,s)\geqslant 0$ and $K(t,s)\geqslant 0, \forall (t,s)\in[a,T]^2$.

— Property (iv). Through direct accounts we can check that $G(t,s)$ is nondecreasing w.r.t. $t\in[a,T]$.

$$\frac{\partial G}{\partial t}(t,s)=\begin{cases}\frac{t^{\rho_1-1}}{\Gamma(\alpha-1)}\left(\frac{T^{\rho_1}-s^{\rho_1}}{\rho_1}\right)^{\alpha-2}s^{\rho_1-1}-\frac{t^{\rho_1-1}}{\Gamma(\alpha-1)}\left(\frac{t^{\rho_1}-s^{\rho_1}}{\rho_1}\right)^{\alpha-2}s^{\rho_1-1}, & a\leqslant s\leqslant t\leqslant T,\\ \frac{t^{\rho_1-1}}{\Gamma(\alpha-1)}\left(\frac{T^{\rho_1}-s^{\rho_1}}{\rho_1}\right)^{\alpha-2}s^{\rho_1-1}, & a\leqslant t\leqslant s\leqslant T.\end{cases}\tag{19}$$

Thus, $G(t,s)$ is increasing with respect to $t\in[a,T]$ and therefore, $G(t,s)\leqslant G(T,s)$, for $a\leqslant t,s\leqslant T$. Furthermore, for $t\leqslant s$, we have

$$\begin{aligned}\frac{\partial K(t,s)}{\partial t}&=-\frac{t^{\rho_2-1}}{\Gamma(\sigma-1)}\left(\frac{s^{\rho_2}-a^{\rho_2}}{\rho_2}\right)^{\sigma-2}s^{\rho_2-1}+\frac{(\sigma-1)t^{\rho_2-1}}{\Gamma(\sigma)}\left(\frac{s^{\rho_2}-t^{\rho_2}}{\rho_2}\right)^{\sigma-2}s^{\rho_2-1},\\&=\frac{t^{\rho_2-1}}{\Gamma(\sigma-1)}s^{\rho_2-1}\left[\left(\frac{s^{\rho_2}-t^{\rho_2}}{\rho_2}\right)^{\sigma-2}-\left(\frac{s^{\rho_2}-a^{\rho_2}}{\rho_2}\right)^{\sigma-2}\right],\\&\leqslant\frac{t^{\rho_2-1}}{\Gamma(\sigma-1)}s^{\rho_2-1}\left[\left(\frac{s^{\rho_2}-a^{\rho_2}}{\rho_2}\right)^{\sigma-2}-\left(\frac{s^{\rho_2}-a^{\rho_2}}{\rho_2}\right)^{\sigma-2}\right]=0.\end{aligned}$$

For $a\leqslant s\leqslant t\leqslant T$, we have

$$\frac{\partial K(t,s)}{\partial t}=\frac{-t^{\rho_2-1}}{\Gamma(\sigma-1)}\left(\frac{s^{\rho_2}-a^{\rho_2}}{\rho_2}\right)^{\sigma-2}s^{\rho_2-1}\leqslant 0.$$

Thus, $K(t,s)$ is nonincreasing with respect to t . Consequently $K(t,s)\leqslant K(a,s)\quad\forall t,s\in[a,T]$. On the other hand, when $t\geqslant s$

$$\begin{aligned}\frac{G(t,s)}{G(T,s)}&=\frac{(\alpha-1)(t^{\rho_1}-a^{\rho_1})(T^{\rho_1}-s^{\rho_1})^{\alpha-2}-(t^{\rho_1}-s^{\rho_1})^{\alpha-1}}{(\alpha-1)(T^{\rho_1}-a^{\rho_1})(T^{\rho_1}-s^{\rho_1})^{\alpha-2}-(T^{\rho_1}-s^{\rho_1})^{\alpha-1}},\\&=\frac{(\alpha-1)(t^{\rho_1}-a^{\rho_1})(T^{\rho_1}-s^{\rho_1})^{\alpha-2}-(t^{\rho_1}-s^{\rho_1})^{\alpha-1}\left(\frac{t^{\rho_1}-s^{\rho_1}}{t^{\rho_1}-a^{\rho_1}}\right)^{\alpha-1}}{(\alpha-1)(T^{\rho_1}-a^{\rho_1})(T^{\rho_1}-s^{\rho_1})^{\alpha-2}-(T^{\rho_1}-s^{\rho_1})^{\alpha-1}}.\end{aligned}$$

As $\left(\frac{t^\rho - s^\rho}{t^\rho - a^\rho}\right)^\beta \leq \left(\frac{T^\rho - s^\rho}{T^\rho - a^\rho}\right)^\beta$ for $\beta > 0$, we obtain

$$\begin{aligned}
\frac{G(t,s)}{G(T,s)} &\geqslant \frac{(\alpha-1)(t^{\rho_1}-a^{\rho_1})(T^{\rho_1}-s^{\rho_1})^{\alpha-2} - (t^{\rho_1}-s^{\rho_1})^{\alpha-1}\left(\frac{T^{\rho_1}-s^{\rho_1}}{T^{\rho_1}-a^{\rho_1}}\right)^{\alpha-1}}{(\alpha-1)(T^{\rho_1}-a^{\rho_1})(T^{\rho_1}-s^{\rho_1})^{\alpha-2} - (T^{\rho_1}-s^{\rho_1})^{\alpha-1}}, \\
&\geqslant \frac{(t^{\rho_1}-a^{\rho_1})^{\alpha-1}}{(T^{\rho_1}-a^{\rho_1})^{\alpha-1}} \frac{(\alpha-1)(T^{\rho_1}-a^{\rho_1})^{\alpha-1}(t^{\rho_1}-a^{\rho_1})^{2-\alpha}(T^{\rho_1}-s^{\rho_1})^{\alpha-2} - (T^{\rho_1}-s^{\rho_1})^{\alpha-1}}{(\alpha-1)(T^{\rho_1}-a^{\rho_1})(T^{\rho_1}-s^{\rho_1})^{\alpha-2} - (T^{\rho_1}-s^{\rho_1})^{\alpha-1}}, \\
&\geqslant \frac{(t^{\rho_1}-a^{\rho_1})^{\alpha-1}}{(T^{\rho_1}-a^{\rho_1})^{\alpha-1}} \frac{\left(\frac{T^{\rho_1}-a^{\rho_1}}{t^{\rho_1}-a^{\rho_1}}\right)^{\alpha-2}(\alpha-1)(T^{\rho_1}-a^{\rho_1})(T^{\rho_1}-s^{\rho_1})^{\alpha-2} - (T^{\rho_1}-s^{\rho_1})^{\alpha-1}}{(\alpha-1)(T^{\rho_1}-a^{\rho_1})(T^{\rho_1}-s^{\rho_1})^{\alpha-2} - (T^{\rho_1}-s^{\rho_1})^{\alpha-1}}, \\
&\geqslant \frac{(t^{\rho_1}-a^{\rho_1})^{\alpha-1}}{(T^{\rho_1}-a^{\rho_1})^{\alpha-1}}.
\end{aligned}$$

For $t \leqslant s$, we have

$$\frac{G(t,s)}{(t^{\rho_1}-a^{\rho_1})^{\alpha-1}} = \frac{\rho_1^{\alpha-1}\, s^{\rho_1-1}}{\Gamma(\alpha-1)}\,(T^{\rho_1}-s^{\rho_1})^{\alpha-2}\,\frac{1}{(t^{\rho_1}-a^{\rho_1})^{\alpha-2}},$$

which is a nonincreasing function as $\alpha \geq 0$. Consequently, $\frac{G(t,s)}{(t^{\rho_1}-a^{\rho_1})^{\alpha-1}} \geq \frac{G(T,s)}{(T^{\rho_1}-a^{\rho_1})^{\alpha-1}}$, which implies $G(t,s) \geqslant \left(\frac{t^{\rho_1}-a^{\rho_1}}{T^{\rho_1}-a^{\rho_1}}\right)^{\alpha-1} G(T,s)$.

Using similar techniques, one can prove that $K(t,s) \geqslant \left(\frac{T^{\rho_2}-t^{\rho_2}}{T^{\rho_2}-a^{\rho_2}}\right)^{\sigma-1} K(a,s)$, for $a \leqslant s,t < T$. Therefore (iv) of Lemma 10 holds.

— Property (v). Again, we can consider two cases. Nevertheless, we prove the results for the case $s \leq t$ only. The simpler case $a \leq t \leq s < T$ can be treated similar arguments.

When $s \leq t$ we have

$$\frac{G_t'(t,s)}{G(T,s)}\,\frac{(T^{\rho_1}-a^{\rho_1})}{t^{\rho_1-1}\rho_1(\alpha-1)} = \frac{\left[(T^{\rho_1}-s^{\rho_1})^{\alpha-2} - (t^{\rho_1}-s^{\rho_1})^{\alpha-2}\right]}{(\alpha-1)\,(T^{\rho_1}-s^{\rho_1})^{\alpha-2} - \frac{(T^{\rho_1}-s^{\rho_1})^{\alpha-1}}{(T^{\rho_1}-a^{\rho_1})}}.$$

Consequently,

$$\begin{aligned}
\frac{G_t'(t,s)}{G(T,s)}\,\frac{(T^{\rho_1}-a^{\rho_1})}{t^{\rho_1-1}\rho_1(\alpha-1)} &\leq \frac{\left[(T^{\rho_1}-s^{\rho_1})^{\alpha-2}\right]}{(\alpha-1)\,(T^{\rho_1}-s^{\rho_1})^{\alpha-2} - \frac{(T^{\rho_1}-s^{\rho_1})^{\alpha-1}}{(T^{\rho_1}-a^{\rho_1})}}, \\
&\leq \frac{1}{(\alpha-1) - \frac{(T^{\rho_1}-s^{\rho_1})}{(T^{\rho_1}-a^{\rho_1})}}, \\
&\leq \frac{1}{(\alpha-2)}.
\end{aligned}$$

Using similar techniques, we get

$$\frac{|\,K_t'(t,s)\,|}{K(a,s)}\,\frac{(T^{\rho_2}-a^{\rho_2})}{t^{\rho_2-1}\rho_2} \leq 1.$$

On the other hand,

$$\begin{aligned}\frac{G_t'(t,s)}{G(T,s)}\,\frac{(T^{\rho_1}-a^{\rho_1})}{t^{\rho_1-1}\rho_1} &= \frac{(\alpha-1)\left[(T^{\rho_1}-s^{\rho_1})^{\alpha-2}-(t^{\rho_1}-s^{\rho_1})^{\alpha-2}\right]}{(\alpha-1)\,(T^{\rho_1}-s^{\rho_1})^{\alpha-2}-\frac{(T^{\rho_1}-s^{\rho_1})^{\alpha-1}}{(T^{\rho_1}-a^{\rho_1})}},\\ &\geq 1-\frac{(t^{\rho_1}-s^{\rho_1})^{\alpha-2}}{(T^{\rho_1}-s^{\rho_1})^{\alpha-2}},\\ &\geq 1-\left(\frac{t}{T}\right)^{\rho_1(\alpha-2)}\left(\frac{1-\left(\frac{s}{t}\right)^{\rho_1}}{1-\left(\frac{s}{T}\right)^{\rho_1}}\right)^{\alpha-2},\\ &\geq 1-\left(\frac{t}{T}\right)^{\rho_1(\alpha-2)}.\end{aligned}$$

Thus, the proof is completed. ■

Now, consider the Banach space $\mathbb{E}=C^3_{\rho_1}(J)$. We can define norm

$$\|u\|=\sum_{k=0}^{3}\max_{a\leq t\leq T}|\delta^k_{\rho_1}u(t)|,$$

and the cone

$$E=\{u\in\mathbb{E}: u \text{ is nonnegative and increasing }\}.$$

Lemma 11 *Let u be the unique solution of the BVP* (15) *associated to a given $g(t)\in C^+(J)$. Then $u\in E$, and the following inequality hold for $t\in[\bar\theta,\theta]\subset(a,T)$.*

$$\min_{t\in[\bar\theta,\theta]}u(t)\geqslant\left(\frac{\bar\theta^{\rho_1}-a^{\rho_1}}{T^{\rho_1}-a^{\rho_1}}\right)^{\alpha-1}\mathrm{M}\,\|u\|, \tag{20}$$

where

$$\mathrm{M}=4\min\left\{1,\frac{\alpha-2}{\alpha-1}\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1}\right),\min\left\{\Gamma(\alpha-1)\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{2-\alpha},\frac{1}{(\bar p-1)\varsigma}\right\}\times\left(\frac{\bar\theta^{\rho_1}-a^{\rho_1}}{T^{\rho_1}-a^{\rho_1}}\right)^{\alpha-1}Z(\theta)\int_a^T G(T,s)\mathrm{d}s,\right\},$$

$Z(t)=\phi_{\bar p}\left(\left(\frac{T^{\rho_2}-t^{\rho_2}}{T^{\rho_2}-a^{\rho_2}}\right)^{\sigma-1}\right)$ *and* $\varsigma=\frac{\rho_2}{(T^{\rho_2}-a^{\rho_2})\Gamma(\alpha-2)}\max_{a\leqslant t\leqslant T}\left\{\int_a^t\left(\frac{s^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha-3}s^{\rho_1-1}\,(t^{\rho_1}+a^{\rho_1}-s^{\rho_1})^{\frac{\rho_2-\rho_1}{\rho_1}}\mathrm{d}s\right\}$.

Proof. From Lemma (9), we have

$$\begin{aligned}u(t)=&\int_a^T G(t,s)\phi_{\bar p}\left(\int_a^T K(s,\tau)g(\tau)\mathrm{d}\tau\right)\mathrm{d}s+\left(\frac{t^{\rho_1}-a^{\rho_1}}{\rho_1-\mu\rho_1}\right)\mu\int_a^T H(\eta,s)\phi_{\bar p}\left(\int_a^T K(s,\tau)g(\tau)\mathrm{d}\tau\right)\mathrm{d}s\\&+\left(\frac{t^{\rho_1}-a^{\rho_1}}{\rho_1-\mu\rho_1}\right)\lambda+\phi_{\bar p}\left(\int_a^T K(a,\tau)g(\tau)\mathrm{d}\tau\right).\end{aligned}$$

(1) The functions G,H, and K are non-negative (Lemma (10)-(*iii*)). Thus, u is also non-negative. Furthermore, as G is increasing w.r.t. t (Lemma (10)-(*iv*), so it is the function u.

(2) As u is non-negative and increasing, we have

$$\begin{aligned}\max_{a\leqslant t\leqslant T}|u(t)|=u(T)=&\int_a^T G(T,s)\phi_{\bar p}\left(\int_a^T K(s,\tau)g(\tau)\mathrm{d}\tau\right)\mathrm{d}s\\&+\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1-\mu\rho_1}\right)\mu\int_a^T H(\eta,s)\phi_{\bar p}\left(\int_a^T K(s,\tau)g(\tau)\mathrm{d}\tau\right)\mathrm{d}s\\&+\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1-\mu\rho_1}\right)\lambda+\phi_{\bar p}\left(\int_a^T K(a,\tau)g(\tau)\mathrm{d}\tau\right).\end{aligned}$$

For $t \in [\bar{\theta}, \theta]$, using (iv) of Lemma 10 and the fact that $\left(\frac{\bar{\theta}^{\rho_1}-a^{\rho_1}}{T^{\rho_1}-a^{\rho_1}}\right) < 1$, we get

$$\begin{aligned} u(t) \geqslant & \int_a^T \left(\frac{\bar{\theta}^{\rho_1}-a^{\rho_1}}{T^{\rho_1}-a^{\rho_1}}\right)^{\alpha-1} G(T,s)\phi_{\bar{p}}\left(\int_a^T K(s,\tau)g(\tau)\mathrm{d}\tau\right)\mathrm{d}s \\ & + \left(\frac{\bar{\theta}^{\rho_1}-a^{\rho_1}}{T^{\rho_1}-a^{\rho_1}}\right)^{\alpha-2} \times \left(\frac{t^{\rho_1}-a^{\rho_1}}{\rho_1-\mu\rho_1}\right)\mu\int_a^T H(\eta,s)\phi_{\bar{p}}\left(\int_a^T K(s,\tau)g(\tau)\mathrm{d}\tau\right)\mathrm{d}s \\ & + \left(\frac{\bar{\theta}^{\rho_1}-a^{\rho_1}}{T^{\rho_1}-a^{\rho_1}}\right)^{\alpha-2} \times \left(\frac{t^{\rho_1}-a^{\rho_1}}{\rho_1-\mu\rho_1}\right)\lambda + \left(\frac{\bar{\theta}^{\rho_1}-a^{\rho_1}}{T^{\rho_1}-a^{\rho_1}}\right)^{\alpha-1} \times \phi_{\bar{p}}\left(\int_a^T K(a,\tau)g(\tau)\mathrm{d}\tau\right). \end{aligned}$$

Consequently,

$$u(t) \geqslant \left(\frac{\bar{\theta}^{\rho_1}-a^{\rho_1}}{T^{\rho_1}-a^{\rho_1}}\right)^{\alpha-1} \max_{a\leqslant t\leqslant T}|u(t)| \tag{21}$$

.

(3) We have

$$\begin{aligned} \delta^1_{\rho_1}u(t) = t^{1-\rho_1}\int_a^T G'_t(t,s)\phi_{\bar{p}}\left(\int_a^T K(s,\tau)g(\tau)d\tau\right)\mathrm{d}s \\ + \frac{\mu}{(1-\mu)}\int_a^T H(\eta,s)\phi_{\bar{p}}\left(\int_a^T K(s,\tau)g(\tau)d\tau\right)\mathrm{d}s + \frac{\lambda}{(1-\mu)}. \end{aligned} \tag{22}$$

From Lemma 10 ((*iii*) and (*v*)), we can deduce that $\delta^1_{\rho_1}u(t) \geq 0$ and

$$\begin{aligned} \delta^1_{\rho_1}u(t) \leqslant & \int_a^T \frac{\alpha-1}{\alpha-2}\frac{\rho_1}{T^{\rho_1}-a^{\rho_1}}G(T,s)\phi_{\bar{p}}\left(\int_a^T K(s,\tau)g(\tau)d\tau\right)\mathrm{d}s \\ & + \frac{\mu}{(1-\mu)}\int_a^T H(\eta,s)\phi_{\bar{p}}\left(\int_a^T K(s,\tau)g(\tau)d\tau\right)\mathrm{d}s + \frac{\lambda}{(1-\mu)}, \\ \leqslant & \frac{\alpha-1}{\alpha-2}\frac{\rho_1}{T^{\rho_1}-a^{\rho_1}} \times \left[\int_a^T G(T,s)\phi_{\bar{p}}\left(\int_a^T K(s,\tau)g(\tau)d\tau\right)\mathrm{d}s\right. \\ & \left. + \mu\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1-\mu\rho_1}\right)\int_a^T H(\eta,s)\phi_{\bar{p}}\left(\int_a^T K(s,\tau)g(\tau)d\tau\right)\mathrm{d}s + \lambda\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1-\mu\rho_1}\right)\right], \\ \leqslant & \frac{\alpha-1}{\alpha-2}\frac{\rho_1}{T^{\rho_1}-a^{\rho_1}} \times \left[\int_a^T G(T,s)\phi_{\bar{p}}\left(\int_a^T K(s,\tau)g(\tau)d\tau\right)\mathrm{d}s + \lambda\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1-\mu\rho_1}\right) + \right. \\ & \left. \mu\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1-\mu\rho_1}\right)\int_a^T H(\eta,s)\phi_{\bar{p}}\left(\int_a^T K(s,\tau)g(\tau)d\tau\right)\mathrm{d}s + \phi_{\bar{p}}\left(\int_a^T K(a,\tau)g(\tau)d\tau\right)\right], \\ \leqslant & \frac{\alpha-1}{\alpha-2}\frac{\rho_1}{T^{\rho_1}-a^{\rho_1}} \times u(T). \end{aligned} \tag{23}$$

.

(4) A straightforward calculus gives

$$\delta^2_{\rho_1}u(t) = -\frac{1}{\Gamma(\alpha-2)}\int_a^t \left(\frac{t^{\rho_1}-s^{\rho_1}}{\rho_1}\right)^{\alpha-3} s^{\rho_1-1}\,\phi_{\bar{p}}\left(\int_a^T K(s,\tau)g(\tau)\mathrm{d}\tau\right)\mathrm{d}s. \tag{24}$$

Then, we get :

$$\begin{aligned} |\delta^2_{\rho_1}u(t)| &\leqslant \phi_{\bar{p}}\left(\int_a^T K(a,\tau)g(\tau)\mathrm{d}\tau\right) \times \frac{1}{\Gamma(\alpha-2)}\int_a^t \left(\frac{t^{\rho_1}-s^{\rho_1}}{\rho_1}\right)^{\alpha-3} s^{\rho_1-1}\mathrm{d}s, \\ &\leqslant \phi_{\bar{p}}\left(\int_a^T K(a,\tau)g(\tau)\mathrm{d}\tau\right) \times \frac{1}{\Gamma(\alpha-1)}\left(\frac{t^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha-2}. \end{aligned}$$

Thus,

$$\max_{a\leqslant t\leqslant T}|\delta^2_{\rho_1}u(t)|\leqslant \phi_{\bar{p}}\left(\int_a^T K(a,\tau)g(\tau)\mathrm{d}\tau\right)\ \frac{1}{\Gamma(\alpha-1)}\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha-2}.$$

By multiplying both sides of the previous inequality by $\phi_{\bar{p}}\left(\left(\frac{T^{\rho_2}-s^{\rho_2}}{T^{\rho_2}-a^{\rho_2}}\right)^{\sigma-1}\right)$, we get

$$\phi_{\bar{p}}\left(\left(\frac{T^{\rho_2}-s^{\rho_2}}{T^{\rho_2}-a^{\rho_2}}\right)^{\sigma-1}\right)\ \max_{a\leqslant t\leqslant T}|\delta^2_{\rho_1}u(t)|\leqslant \phi_{\bar{p}}\left(\int_a^T\left(\frac{T^{\rho_2}-s^{\rho_2}}{T^{\rho_2}-a^{\rho_2}}\right)^{\sigma-1}K(a,\tau)g(\tau)\mathrm{d}\tau\right)\ \frac{1}{\Gamma(\alpha-1)}\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha-2},$$

using Lemma (10)-(iv), we get :

$$\phi_{\bar{p}}\left(\left(\frac{T^{\rho_2}-s^{\rho_2}}{T^{\rho_2}-a^{\rho_2}}\right)^{\sigma-1}\right)\ \max_{a\leqslant t\leqslant T}|\delta^2_{\rho_1}u(t)|\leqslant \phi_{\bar{p}}\left(\int_a^T K(s,\tau)g(\tau)\mathrm{d}\tau\right)\ \frac{1}{\Gamma(\alpha-1)}\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha-2}. \tag{25}$$

We multiplying both sides by $G(t,s)$ and integrate over $[a,T]$ w.r.t. s, we get

$$\begin{aligned}
&\max_{a\leqslant t\leqslant T}|\delta^2_{\rho_1}u(t)|\ \int_a^T G(t,s)\phi_{\bar{p}}\left(\left(\frac{T^{\rho_2}-s^{\rho_2}}{T^{\rho_2}-a^{\rho_2}}\right)^{\sigma-1}\right)\mathrm{d}s\\
&\qquad\leqslant \frac{1}{\Gamma(\alpha-1)}\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha-2}\ \int_a^T G(t,s)\phi_{\bar{p}}\left(\int_a^T K(s,\tau)g(\tau)\mathrm{d}\tau\right)\mathrm{d}s,\\
&\qquad\leqslant \frac{1}{\Gamma(\alpha-1)}\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha-2}\ \left[\int_a^T G(t,s)\phi_{\bar{p}}\left(\int_a^T K(s,\tau)g(\tau)\mathrm{d}\tau\right)\mathrm{d}s+\left(\frac{t^{\rho_1}-a^{\rho_1}}{\rho_1-\mu\rho_1}\right)\lambda\right.\\
&\qquad\left.\left(\frac{t^{\rho_1}-a^{\rho_1}}{\rho_1-\mu\rho_1}\right)\mu\int_a^T H(\eta,s)\phi_{\bar{p}}\left(\int_a^T K(s,\tau)g(\tau)\mathrm{d}\tau\right)\mathrm{d}s+\phi_{\bar{p}}\left(\int_a^T K(a,\tau)g(\tau)\mathrm{d}\tau\right)\right],\\
&\qquad=:\frac{1}{\Gamma(\alpha-1)}\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha-2}\ u(t),\\
&\qquad\leqslant \frac{1}{\Gamma(\alpha-1)}\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha-2}\ \max_{a\leqslant t\leqslant T}|u(t)|
\end{aligned}$$

Furthermore, for $t\in[\bar{\theta},\theta]$

$$\int_a^T G(t,s)\phi_{\bar{p}}\left(\left(\frac{T^{\rho_2}-s^{\rho_2}}{T^{\rho_2}-a^{\rho_2}}\right)^{\sigma-1}\right)\mathrm{d}s\geqslant\left(\frac{t^{\rho_1}-a^{\rho_1}}{T^{\rho_1}-a^{\rho_1}}\right)^{\alpha-1}Z(\theta)\int_a^T G(T,s)\mathrm{d}s,$$

$$\begin{aligned}
&\max_{a\leqslant t\leqslant T}|\delta^2_{\rho_1}u(t)|\ \int_a^T G(t,s)\phi_{\bar{p}}\left(\left(\frac{T^{\rho_2}-s^{\rho_2}}{T^{\rho_2}-a^{\rho_2}}\right)^{\sigma-1}\right)\mathrm{d}s\geqslant\\
&\qquad\left(\frac{t^{\rho_1}-a^{\rho_1}}{T^{\rho_1}-a^{\rho_1}}\right)^{\alpha-1}Z(\theta)\int_a^T G(T,s)\mathrm{d}s\ \max_{a\leqslant t\leqslant T}|\delta^2_{\rho_1}u(t)|,
\end{aligned}\tag{26}$$

.

(5) From the equation in (24) and Remark 1 , one can see that

$$\delta^2_{\rho_1}u(t)=-\frac{1}{\Gamma(\alpha-1)}\int_a^t\left(\frac{s^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha-2}s^{\rho_1-1}\,\phi_{\bar{p}}\left(\int_a^T K\left((t^{\rho_1}+a^{\rho_1}-s^{\rho_1})^{\frac{1}{\rho_1}},\tau\right)g(\tau)\mathrm{d}\tau\right)\mathrm{d}s. \tag{27}$$

Then, from (27) and Lemma 4 we get

$$\delta^3_{\rho_1}u(t)=\delta^1_{\rho_1}\left(\delta^2_{\rho_1}u\right)(t)=-\frac{t^{1-\rho_1}}{\Gamma(\alpha-2)}\int_a^t\left(\frac{s^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha-3}s^{\rho_1-1}\phi_{\bar p}\left(\int_a^T K\left((t^{\rho_1}+a^{\rho_1}-s^{\rho_1})^{\frac{1}{\rho_1}},\tau\right)' g(\tau)\mathrm{d}\tau\right)\mathrm{d}s,$$

where

$$\phi_{\bar p}\left(\int_a^T K\left((t^{\rho_1}+a^{\rho_1}-s^{\rho_1})^{\frac{1}{\rho_1}},\tau\right)g(\tau)\mathrm{d}\tau\right)'=\int_a^T K'_t\left((t^{\rho_1}+a^{\rho_1}-s^{\rho_1})^{\frac{1}{\rho_1}},\tau\right)g(\tau)\mathrm{d}\tau\times$$
$$\frac{(\bar p-1)t^{\rho_1-1}}{(t^{\rho_1}+a^{\rho_1}-s^{\rho_1})^{\frac{\rho_1-1}{\rho_1}}}\left|\int_a^T K\left((t^{\rho_1}+a^{\rho_1}-s^{\rho_1})^{\frac{1}{\rho_1}},\tau\right)g(\tau)\mathrm{d}\tau\right|^{p-2}. \quad (28)$$

Thus,

$$\begin{aligned}
|\,\delta^3_{\rho_1}u(t)\,| &= \frac{-1}{\Gamma(\alpha-2)}\int_a^t\left(\frac{s^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha-3}s^{\rho_1-1}\int_a^T K'_t\left((t^{\rho_1}+a^{\rho_1}-s^{\rho_1})^{\frac{1}{\rho_1}},\tau\right)g(\tau)\mathrm{d}\tau\times\\
&\qquad\frac{(\bar p-1)}{(t^{\rho_1}+a^{\rho_1}-s^{\rho_1})^{\frac{\rho_1-1}{\rho_1}}}\left|\int_a^T K\left((t^{\rho_1}+a^{\rho_1}-s^{\rho_1})^{\frac{1}{\rho_1}},\tau\right)g(\tau)\mathrm{d}\tau\right|^{p-2}\mathrm{d}s,\\
&\leqslant\frac{\rho_2}{(T^{\rho_2}-a^{\rho_2})\Gamma(\alpha-2)}\int_a^t\left(\frac{s^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha-3}s^{\rho_1-1}\,(t^{\rho_1}+a^{\rho_1}-s^{\rho_1})^{\frac{\rho_2-1}{\rho_1}}\int_a^T K(a,\tau)\,g(\tau)\mathrm{d}\tau\times\\
&\frac{(\bar p-1)}{(t^{\rho_1}+a^{\rho_1}-s^{\rho_1})^{\frac{\rho_1-1}{\rho_1}}}\left|\int_a^T K(a,\tau)g(\tau)\mathrm{d}\tau\right|^{p-2}\mathrm{d}s\\
&=(\bar p-1)\phi_{\bar p}\left(\int_a^T K(a,\tau)g(\tau)\mathrm{d}\tau\right)\frac{\rho_2}{(T^{\rho_2}-a^{\rho_2})\Gamma(\alpha-2)}\int_a^t\left(\frac{s^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha-3}s^{\rho_1-1}\,(t^{\rho_1}+a^{\rho_1}-s^{\rho_1})^{\frac{\rho_2-\rho_1}{\rho_1}}\mathrm{d}s\\
&\leqslant(\bar p-1)\phi_{\bar p}\left(\int_a^T K(a,\tau)g(\tau)\mathrm{d}\tau\right)\frac{\rho_2}{(T^{\rho_2}-a^{\rho_2})\Gamma(\alpha-2)}\max_{a\leqslant t\leqslant T}\left\{\int_a^t\left(\frac{s^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha-3}s^{\rho_1-1}\,(t^{\rho_1}+a^{\rho_1}-s^{\rho_1})^{\frac{\rho_2-\rho_1}{\rho_1}}\mathrm{d}s\right\},
\end{aligned}$$

Then,

$$\max_{a\leqslant t\leqslant T}|\delta^3_{\rho_1}u(t)|\leqslant(\bar p-1)\phi_{\bar p}\left(\int_a^T K(a,\tau)g(\tau)\mathrm{d}\tau\right)\times\varsigma. \quad (29)$$

As in (2).

(6) From the inequalities (21), (23), (26) and (29) one can see that, equation(20) is a direct consequence of the previous results.

■

Then, for selected $[\bar\theta,\theta]\subset(a,T)$, we define the cone

$$\mathscr{P}=\left\{u\in E:\min_{t\in[\bar\theta,\theta]}u(t)\geqslant\left(\frac{\bar\theta^{\rho_1}-a^{\rho_1}}{T^{\rho_1}-a^{\rho_1}}\right)^{\alpha-1}\mathrm{M}\|u\|\right\},$$

and the integral operator $\mathscr{N}_\lambda:\mathscr{P}\to\mathbb{E}$ defined for $t\in[\bar\theta,\theta]$ by

$$\begin{aligned}
\mathscr{F}_\lambda(u)(t)&=\int_a^T G(t,s)\phi_{\bar p}\left(\int_a^T K(s,\tau)cf(u(\tau))\,d\tau\right)\mathrm{d}s\\
&\quad+\mu\left(\frac{t^{\rho_1}-a^{\rho_1}}{\rho_1-\mu\rho_1}\right)\int_a^T H(\eta,s)\,\phi_{\bar p}\left(\int_a^T K(s,\tau)cf(u(\tau))\,d\tau\right)\mathrm{d}s\\
&\quad+\left(\frac{t^{\rho_1}-a^{\rho_1}}{\rho_1-\mu\rho_1}\right)\lambda+\phi_{\bar p}\left(\int_a^T K(a,\tau)cf(u(\tau))\,d\tau\right). \quad (30)
\end{aligned}$$

We have $\mathscr{F}_\lambda(\mathscr{P}) \subset \mathscr{P}$ and fixed points of $\mathscr{F}_\lambda$ are solutions of (1). To use some fixed point Theorems, we need to show that $\mathscr{N}_\lambda$ is completely continuous.

Lemma 12 *[1] Let $h,s>0$. For any $x,y\in[0,c]$, the following propositions hold.*

1 If $s>1$, then $|x^s-y^s|\leqslant sh^{s-1}|x-y|$.

2 If $0<s\leqslant 1$, then $|x^s-y^s|\leqslant|x-y|^s$.

Lemma 13 *The integral operator $\mathscr{F}_\lambda:\mathscr{C}\to\mathscr{C}$ is continuous and compact.*

Proof.

The continuity of $\mathscr{F}_\lambda$ is a consequence of the continuity and positiveness of G,K,c and f.

To prove that $\mathscr{F}_\lambda$ is compact, let us consider a bounded subset $\Omega\subset\mathscr{P}$. Then, there exists $L>0$ such that for any $u\in\Omega$ we have

$$|f(u(t))|\leqslant L.$$

For any $u\in\Omega$, as $\mathscr{F}_\lambda(u)$ is positive , continuous and G is increasing w.r.t. t, The maximum is at the value $t=T$ i.e.,

$$\max_{a\leqslant t\leqslant T}|\,\mathscr{F}_\lambda(u(t))\,|=\mathscr{F}_\lambda(u(T)).$$

Consequently, from the above inequality, we get

$$\begin{aligned}\max_{a\leqslant t\leqslant T}|\,\mathscr{F}_\lambda(u(t))\,|\leq&\phi_{\bar p}\left(\int_a^T K(a,\tau)cLd\tau\right)\times\left[\int_a^T G(T,s)\mathrm{d}s+\mu\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1-\mu\rho_1}\right)\int_a^T H(\eta,s)\mathrm{d}s+1\right]\\&+\lambda\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1-\mu\rho_1}\right)\qquad\doteq\qquad\bar L.\end{aligned}\tag{31}$$

Then, as in Lemma (11), we obtain

$$\|\mathscr{F}_\lambda u\|\leqslant\mathbf{N}\,\bar L,$$

where $\mathbf{N}=4\max\left\{1,\frac{\alpha-1}{\alpha-2}\left(\frac{\rho_1}{T^{\rho_1}-a^{\rho_1}}\right),\max\left\{\frac{1}{\Gamma(\alpha-1)}\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha-2},1\right\}\times\left(\left(\frac{\bar\theta^{\rho_1}-a^{\rho_1}}{T^{\rho_1}-a^{\rho_1}}\right)^{\alpha-1}Z(\theta)\int_a^T G(T,s)\mathrm{d}s\right)^{-1}\right\}$.

Hence, $\mathscr{F}_\lambda(\Omega)$ is uniformly bounded. Furthermore, by using Lemmas (9), (12), (10) and Lebesgue dominated convergence theorem, we deduce the equicontinuouty of $\mathscr{F}_\lambda(\Omega)$. Therefore, $\mathscr{F}_\lambda$ is completely continuous by Arzela-Ascoli Theorem. ■

In order to avoid repetition in the remaining, we use the following notations.

$$f_0:=\lim_{r\to0^+}\frac{f(r)}{\phi_p(r)},\ f_\infty=\lim_{r\to\infty}\frac{f(r)}{\phi_p(r)}.$$

$$\gamma:=\left(\frac{\bar\theta^{\rho_1}-a^{\rho_1}}{T^{\rho_1}-a^{\rho_1}}\right)^{\alpha-1}\mathrm{M},$$

$$\Lambda_1:=\left[\left(\int_a^T G(T,s)\mathrm{d}s+\mu\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1-\rho_1\mu}\right)\int_a^T H(\eta,s)\mathrm{d}s+1\right)\times\phi_{\bar p}\left(\int_a^T K(a,\tau)cd\tau\right)\right]^{-1},$$

$$\Lambda_2:=\left[\frac{\alpha-1}{\alpha-2}\frac{\rho_1}{T^{\rho_1}-a^{\rho_1}}\left(\int_a^T G(T,s)\mathrm{d}s+\frac{\mu}{(1-\mu)}\int_a^T H(\eta,s)\mathrm{d}s\right)\times\phi_{\bar p}\left(\int_a^T K(a,\tau)cd\tau\right)\right]^{-1},$$

$$\Lambda_3:=\left[\frac{1}{\Gamma(\alpha-1)}\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1}\right)^{\alpha-2}\times\phi_{\bar p}\left(\int_a^T K(a,\tau)cd\tau\right)\right]^{-1},\quad\Lambda_4:=\left[((\bar p-1)\varsigma\phi_{\bar p}\left(\int_a^T K(a,\tau)cd\tau\right)\right]^{-1},$$

$$\Lambda_5:=\left[\gamma\left(\frac{\bar\theta^{\rho_1}-a^{\rho_1}}{T^{\rho_1}-a^{\rho_1}}\right)^{\alpha-1}\min_{a\leq s\leq\theta}(Z(s))\left(\int_a^T G(T,s)+\mu\left(\frac{T^{\rho_1}-a^{\rho_1}}{\rho_1-\mu\rho_1}\right)H(\eta,s)\mathrm{d}s\right)\times\phi_{\bar p}\left(\int_a^T K(a,\tau)cd\tau\right)\right]^{-1},$$

4. STUDY THE EXISTENCE OF SOLUTIONS IN A CONE

In this section, we set some hypotheses to derive an interval for λ which ensures the existence of positive solutions of the BVP (1).

(H1) There exist $R_1 > 0$ and $m_1 \in (0, \min\left\{\frac{\Lambda_1}{8}, \frac{\Lambda_2}{8}, \frac{\Lambda_3}{4}, \frac{\Lambda_4}{4}\right\})$ such that

$$f(r) \leqslant \phi_p(m_1 R_1), \quad \forall r \in [0, R_1],$$

(H2) There exist $R_2 > 0$ and $m_2 \in (\Lambda_5, \infty)$ such that

$$f(r) \geqslant \phi_p(m_2 R_2), \quad \forall r \in [\gamma R_2, R_2],$$

(H3)

$$f_0 = \omega\prime \in [0, k^{p-1}), \quad k = \frac{\min\left\{\frac{\Lambda_1}{8}, \frac{\Lambda_2}{8}, \frac{\Lambda_3}{4}, \frac{\Lambda_4}{4}\right\}}{4}$$

(H4)

$$f_\infty = \xi \in (\left(\frac{2\Lambda_5}{\gamma}\right)^{p-1}, \infty).$$

Theorem 14 *Assume that the conditions **(H1)**and **(H2)** hold, and $0 < R_1 < R_2$. Then, the BVP (1) has at least one positive solution for $\lambda > 0$ small enough.*

Proof. Let $\mathscr{P}_1 = \{u \in E : \|u\| \leq R_1\}$ and λ satisfying

$$0 < \lambda \leqslant \frac{(1-\mu)R_1}{8} \min\left(1, \frac{\rho_1}{T^{\rho_1} - a^{\rho_1}}\right). \tag{32}$$

Let $u \in E \cap \partial\mathscr{P}_1$, i.e., $\|u\| = R_1$. From (**H1**) and (32), we have

$$\begin{aligned}\max_{a\leqslant t\leqslant T} |\, \mathscr{F}_\lambda(u(t))\,| = \mathscr{F}_\lambda(u(T)) &= \int_a^T G(T,s)\phi_{\bar{p}}\left(\int_a^T K(s,\tau)cf(u(\tau))\,d\tau\right)\mathrm{d}s \\ &+ \mu\left(\frac{T^{\rho_1} - a^{\rho_1}}{\rho_1 - \mu\rho_1}\right)\int_a^T H(\eta,s)\,\phi_{\bar{p}}\left(\int_a^T K(s,\tau)cf(u(\tau))\,d\tau\right)\mathrm{d}s \\ &+ \left(\frac{T^{\rho_1} - a^{\rho_1}}{\rho_1 - \mu\rho_1}\right)\lambda + \phi_{\bar{p}}\left(\int_a^T K(a,\tau)cf(u(\tau))\,d\tau\right).\end{aligned}$$

Then,

$$\max_{a\leqslant t\leqslant T} |\, \mathscr{F}_\lambda(u(t))\,| \leqslant \frac{\Lambda_1 R_1}{8}\left[\left(\int_a^T G(T,s)\mathrm{d}s + \mu\left(\frac{T^{\rho_1} - a^{\rho_1}}{\rho_1 - \rho_1}\right)\int_a^T H(\eta,s)\mathrm{d}s + 1\right) \times \phi_{\bar{p}}\left(\int_a^T K(a,\tau)cd\tau\right)\right] + \frac{R_1}{8}.$$

Consequently,

$$\max_{a\leqslant t\leqslant T} |\, \mathscr{F}_\lambda(u(t))\,| \leqslant \frac{R_1}{8} + \frac{R_1}{8} = \frac{\|u\|}{4}.$$

Similarly, we obtain

$$\sum_{k=1}^{3} \max_{a\leqslant t\leqslant T} |\, \delta^k_{\rho_1}\mathscr{F}_\lambda(u(t))\,| \leqslant \frac{3\|u\|}{4}.$$

Therefore, we conclude that

$$\|\mathscr{F}_\lambda u\| \leqslant \|u\|, \quad \forall u \in E \cap \partial \mathscr{P}_1. \tag{33}$$

On the other hand, we set $\mathscr{P}_2 = \{u \in E : \|u\| \leqslant R_2\}$. Then, for any $u \in E \cap \partial \mathscr{P}_2$ by Lemma (11) one has

$$R_2 \geqslant \min_{\tau \in [\bar{\theta}, \theta]} u(\tau) \geqslant \gamma R_2.$$

Using hypothesis **H2**, we get

$$\begin{aligned}
\mathscr{F}_\lambda(u(T)) \quad &\geqslant \left(\frac{\bar{\theta}^{\rho_1} - a^{\rho_1}}{T^{\rho_1} - a^{\rho_1}}\right)^{\alpha-1} \left[\int_a^T G(T,s)\phi_{\bar{p}}\left(\int_a^T K(s,\tau)cf(u(\tau))\,d\tau\right)\mathrm{d}s + \mu\left(\frac{T^{\rho_1} - a^{\rho_1}}{\rho_1 - \mu\rho_1}\right)\int_a^T H(\eta,s)\right. \\
&\quad \left.\phi_{\bar{p}}\left(\int_a^T K(s,\tau)cf(u(\tau))\,d\tau\right)\mathrm{d}s + \left(\frac{T^{\rho_1} - a^{\rho_1}}{\rho_1 - \mu\rho_1}\right)\lambda + \phi_{\bar{p}}\left(\int_a^T K(a,\tau)cf(u(\tau))\,d\tau\right)\right] \\
&\geqslant \left(\frac{\bar{\theta}^{\rho_1} - a^{\rho_1}}{T^{\rho_1} - a^{\rho_1}}\right)^{\alpha-1} m_2 R_2 \gamma Z(\theta)\left(\int_a^T G(T,s) + \mu\left(\frac{T^{\rho_1} - a^{\rho_1}}{\rho_1 - \mu\rho_1}\right)H(\eta,s)\mathrm{d}s\right) \times \phi_{\bar{p}}\left(\int_a^T K(a,\tau)cd\tau\right) \\
&\geqslant \left(\frac{\bar{\theta}^{\rho_1} - a^{\rho_1}}{T^{\rho_1} - a^{\rho_1}}\right)^{\alpha-1} \Lambda_5 R_2 \gamma Z(\theta)\left(\int_a^T G(T,s) + \mu\left(\frac{T^{\rho_1} - a^{\rho_1}}{\rho_1 - \mu\rho_1}\right)H(\eta,s)\mathrm{d}s\right) \times \phi_{\bar{p}}\left(\int_a^T K(a,\tau)cd\tau\right) \\
&:= R_2 = \|u\|.
\end{aligned}$$

Which implies that

$$\|\mathscr{F}_\lambda u\| \geqslant \|u\|, \quad \text{for any} \quad u \in E \cap \partial \mathscr{P}_2. \tag{34}$$

Therfore, from (33), (34), $R_1 < R_2$ and by applying the first part of Theorem (6), we deduce that, the operator $\mathscr{F}_\lambda$ has at least one fixed point $u \in E \cap \overline{\mathscr{P}_2} \setminus \mathscr{P}_1$, which is positive solution of BVP (1).
■

Theorem 15 *Assume that all conditions **(H1)**, **(H2)** hold. Then, the BVP* (1) *has no positive solution for* λ *large enough.*

Proof. Fome Technical Contradiction, we get the desired result ■

Remark 2

If $f_0 = 0, \ \ f_\infty = \infty$. hold, then the condotions **(H1)** and **(H2)** hold respectively. Moreover, if the function f is nondecreasing, the following theorem holds.

Theorem 16 *Assume that the hypotheses of Theorem* (14) *hold and that f is nondecreasing. Then, there exists $\lambda^* > 0$ such that the BVP* (1) *has at least one positive solution for $\lambda \in (0, \lambda^*)$ and has no positive solution for $\lambda \in (\lambda^*, \infty)$.*

Proof.

Let $\Theta = \{\lambda, \ \text{the BVP (1) has at least one positive solution}\} \subset \mathbb{R}_+^*$ and $\lambda^* = \sup\Theta$. It follows from Theorem (14) that $\Theta \neq \emptyset$ and thus λ^* exists. We denote by u_0 the solution of BVP (1) associated to λ_0 and

$$\mathscr{K}(u_0) = \{u \in E : u(t) < u_0(t), \forall t \in [a,T]\}.$$

Let $\lambda \in (0, \lambda_0)$ and $u \in \mathscr{K}(u_0)$. It follows from the definition of $\mathscr{N}_\lambda$ (30) and the monotonicity of f that for any t

$$\mathscr{F}_\lambda(u(t)) \leq \mathscr{F}_\lambda(u_0(t)) = u_0(t).$$

Thus, $\mathscr{F}_\lambda(\mathscr{K}(u_0)) \subseteq \mathscr{K}(u_0)$. By Shaulder's fixed point theorem we know that, there exists a fixed point $u \in \mathscr{K}(u_0)$, which is a positive solution of (1). The proof is completed. ■

Theorem 17 *Suppose that f satisfies **(H3)** and **(H4)**. Then, the BVP* (1) *has at least one positive solution forfor $\lambda > 0$ small enough.*

Proof. Firstly, from the definition of f_0, for all $\varepsilon > 0$ there exists an adequate small positive number $\bar{\delta}(\varepsilon)$ such that

$$f(r) \leqslant (\varepsilon + \omega\prime) r^{p-1} \leqslant (\varepsilon + \omega\prime)\bar{\delta}(\varepsilon)^{p-1}, \quad \forall r \in [0, \bar{\delta}(\varepsilon)].$$

Then, for $\varepsilon = k^{p-1} - \omega\prime$ we have

$$f(r) \leqslant k^{p-1}\bar{\delta}(\varepsilon)^{p-1} \leqslant \left(2k\bar{\delta}(\varepsilon)\right)^{p-1}. \tag{35}$$

It's enough to take $R_1 = \bar{\delta}(\varepsilon)$ and $m_1 = 2k \in (0, \min\left\{\frac{\Lambda_1}{8}, \frac{\Lambda_2}{8}, \frac{\Lambda_3}{4}, \frac{\Lambda_4}{4}\right\})$, i.e., the conditions **(H1)** holds.

Next, since $f_\infty = \xi \in (\left(\frac{2\Lambda_5}{\gamma}\right)^{p-1}, \infty)$, then for every $\varepsilon > 0$, there exists an adequate big positive number $R_2 \neq R_1$ such that

$$f(r) \geqslant (\xi - \varepsilon) r^{p-1} \geqslant (\xi - \varepsilon)(\gamma R_2)^{p-1}, \quad \forall r \geqslant \gamma R_2.$$

Hence, for $\varepsilon = \xi - \left(\frac{2\Lambda_5}{\gamma}\right)^{p-1}$ we get

$$f(r) \geqslant \left(\frac{2\Lambda_5}{\gamma}\right)^{p-1} (\gamma R_2)^{p-1} = (2\Lambda_5 R_2)^{p-1}. \tag{36}$$

Let $m_2 = 2\Lambda_5 > \Lambda_5$, thus condition **(H2)** holds, by applying Theorem 14, we complete the proof. ■

5. AN APPLICATIONS

In this section, we give an example to illustrate the usefulness of our main results.

Example 1 *Let us consider the following fractional BVP*

$$\begin{cases} {}^{C-H}\mathscr{D}^{\frac{5}{2}}_{e^{2-}}\left(\phi_p\left({}^{C-H}\mathscr{D}^{\frac{5}{2}}_{e^{1+}}u\right)\right)(t) + \frac{5\sqrt{\pi}}{2^{\frac{7}{2}}}u^{\frac{3}{2}}(t) = 0, & e^1 < t < e^2, \\ u(e^1) + {}^{C-H}\mathscr{D}^{\frac{5}{2}}_{e^{1+}}u(e^1) = 0,\ \delta^2_{0^+}u(e^1) = 0,\ \delta^1_{0^+}u(e^2) = \frac{1}{2}\delta^1_{0^+}u(e^{\frac{8}{5}}) + \lambda, \\ {}^{C-H}\mathscr{D}^{\frac{5}{2}}_{e^{1+}}u(e^2) = 0,\ -\delta^1_{0^+}\left(\phi_p\left({}^{C-H}\mathscr{D}^{\frac{5}{2}}_{e^{1+}}u\right)\right)(e^1) = 0,\ \delta^2_{0^+}\left(\phi_p\left({}^{C-H}\mathscr{D}^{\frac{5}{2}}_{e^{1+}}u\right)\right)(e^2) = 0. \end{cases} \tag{37}$$

Here

$$\begin{array}{ccc} \rho_1 = \rho_2 = 0, & \alpha = \sigma = \frac{5}{2}, & \mu = \frac{1}{2} \\ \eta = e^{\frac{8}{5}}, & p = \frac{3}{2}, & \bar{p} = 3 \\ \theta = e^{\frac{7}{4}}, & \bar{\theta} = e^{\frac{3}{2}}, c = \frac{5\sqrt{\pi}}{2^{\frac{7}{2}}}, & \end{array}$$

${}^{C-H}\mathscr{D}^{\frac{5}{2}}_{e^{1+}}$ *and* ${}^{C-H}\mathscr{D}^{\frac{5}{2}}_{e^{2-}}$ *are the left and right-sided Caputo-Hadamard fractional derivatives.*

We can easily show that $f(u(t)) = u^{\frac{3}{2}}(t)$ satisfy :

$$f_0 = \lim_{u\to 0^+}\frac{f(u)}{\phi_{\frac{3}{2}}(u)} = \lim_{u\to 0^+} u(t) = 0, \quad f_\infty = \lim_{u\to\infty}\frac{f(u)}{\phi_{\frac{3}{2}}(u)} = \lim_{u\to 0^+} u(t) = \infty.$$

Then obviously,

$$\left(\int_{e^1}^{e^2} K(e^1,\tau)c\mathrm{d}\tau\right)^2 = 1,\ \min\left\{\frac{\Lambda_1}{8},\ \frac{\Lambda_2}{8},\ \frac{\Lambda_3}{4},\ \frac{\Lambda_4}{4}\right\} \simeq 0.014589936,\ \ \Lambda_5 \simeq 15.13154296.$$

So, all conditions of Theorem 14 hold, then we can choose $R_2 > R_1$ *and for* λ *satisfies* $0 < \lambda \leqslant \frac{1}{32}R_1$*, such that*

$$\mathscr{P}_1 = \{u \in E : \|u\| < R_1\}, \mathscr{P}_2 = \{u \in E : \|u\| < R_2\}.$$

Then, we can show that, the BVP (37) has at least one positive solution $u \in E \cap (\overline{\mathscr{P}_2} \setminus \mathscr{P}_1)$ *for* λ *small enough.*

6. CONCLUSION

In this words we have discussed the existence and the uniqueness of solutions for a class of nonlinear fractional differential equations with a boundary value, by using the properties of the Green functions associated to (1), the Guo-Krasnosel'skii and Banach fixed point theorems.

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