STUDY OF SOME NON-AUTONOMOUS ABSTRACT PROBLEMS OF ELLIPTIC TYPE IN AN UNBOUNDED DOMAIN

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ABSTRACT

In this work, we are interested to the study of some non-autonomous Dirichlet boundary value problems in an unbounded domain. In the framework of Hölderian spaces, by considering some differentiability assumptions on the resolvents of square roots of linear operators and under some compatibility conditions, we will prove main results on the existence, uniqueness and maximal regularity of the classical solution of this kind of problems which have not been studied in variable coefficients case. The approach used here is based on the fractional powers of linear operators, the semigroups theory, the Dunford's functional calculus and the interpolation spaces.

1. INTRODUCTION

Let us consider some partial differential equations of elliptic type such as the following model :

$$\frac{\partial^2 u}{\partial x^2}(x,y)+b(x,y)\frac{\partial u}{\partial x}(x,y)+a(x,y)\frac{\partial^2 u}{\partial y^2}(x,y)-\lambda u(x,y)=f(x,y),\ x>0,\ y\in]c,d[,$$

a being positive, a and b having Hölderian regularity with respect to x and some other with respect to y. Under various boundary Dirichlet-Neumann conditions with respect to the variable y (depending on x) and with data

$$\begin{cases} u(0,y)=\varphi(y), \\ u(+\infty,y)=0. \end{cases}$$

This class of PDE's can be written, in some complex Banach space X, as the following complete abstract second order differential equation with variable operator coefficients :

$$u''(x)+B(x)u'(x)+A(x)u(x)-\lambda u(x)=f(x), \quad x>0, \tag{1}$$

under Dirichlet boundary conditions :

$$\begin{cases} u(0)=\varphi, \\ u(+\infty)=0. \end{cases} \tag{2}$$

Here λ is a positive real number, φ is a given element in X, $f\in C_\infty^\theta([0,\infty);X)$, $0<\theta<1$, where $C_\infty^\theta([0,\infty);X)$ is the space of bounded and θ-Hölder continuous-vector valued functions $\phi:[0,\infty)\to X$ such that

$$\sup_{x\geq 0}\|\phi(x)\|_X<+\infty,$$

$$\exists C>0:\forall x,\ s\geq 0,\ \|\phi(x)-\phi(s)\|_X\leq C|x-s|^\theta,$$

$$\text{with } \lim_{x\to+\infty}\phi(x)=0,$$

endowed with the norm

$$\|\phi\|_{C^{\theta}_{\infty}([0,\infty);X)} := \sup_{x\geq 0}\|\phi(x)\|_X + \sup_{x\neq s}\frac{\|\phi(x)-\phi(s)\|_X}{|x-s|^{\theta}}. \tag{3}$$

$(B(x))_{x\geq 0}$ is a family of bounded linear operators, and $(A(x))_{x\geq 0}$ is a family of closed linear operators in X, with domains $D(A(x))$ not necessarily dense in X.
We seek for a classical solution $u(\cdot)$ to Problem (1)-(2), that is :

$$\begin{cases} u\in C^2_{\infty}([0,\infty),X),\ u(x)\in D(A_{\lambda}(x)) \text{ for every } x\geq 0, \\ x\mapsto A_{\lambda}(x)u(x)\in C_{\infty}([0,\infty),X), \\ and \text{ u } satisfies\ Problem\ (1)-(2). \end{cases}$$

2. HYPOTHESES AND MAIN RESULT

We suppose that the family of operators $(B(x))_{x\geq 0}$ satisfies

$$\exists C>0 : \forall x\in[0,\infty),\ \|B(x)\|_{L(X)}\leq C. \tag{4}$$

The term $B(x)u'(x)$ is considered as a "perturbation" in some sense.
Setting $A_{\lambda}(x)=A(x)-\lambda I,\ \lambda>0$. The main hypothesis of this work is :

$$\begin{cases} \exists C>0,\ \forall z\geq 0,\ \forall x\geq 0,\ \exists (A_{\lambda}(x)-zI)^{-1}\in\mathscr{L}(X), \\ \left\|(A_{\lambda}(x)-zI)^{-1}\right\|_{\mathscr{L}(X)}\leq \dfrac{C}{(1+z)}. \end{cases} \tag{5}$$

Thanks to this Assumption, for every $x\geq 0$ and $\lambda>0$, the square roots :

$$K_{\lambda}(x) = -(-A_{\lambda}(x))^{1/2}$$

are well defined and generate analytic semigroups $(e^{yK_{\lambda}(x)})_{y>0}$ not necessarily strongly continuous in 0, see [2] for dense domains and [6] for non dense domains.
For all $x\geq 0$, $y>0$, the semigroups $e^{yK_{\lambda}(x)}$ are defined by :

$$e^{yK_{\lambda}(x)} = -\frac{1}{2i\pi}\int_{\Gamma} e^{yz}(K_{\lambda}(x)-zI)^{-1}dz,$$

where the curve Γ is the boundary of a sector : $S_{\theta_1+\pi/2,r_1}\subset\rho(K_{\lambda}(x))$ defined by

$$S_{\theta_1+\pi/2,r_1} = \{z\in\mathbb{C}\backslash\{0\} : |\arg(z)|\leq\theta_1+\pi/2\}\cup\{z\in\mathbb{C} : |z|\leq r_1\},$$

oriented from $\infty e^{-i(\theta_1+\pi/2)}$ to $\infty e^{i(\theta_1+\pi/2)}$ and for all $z\geq 0$, $x\geq 0$, we have

$$(K_{\lambda}(x)-zI)^{-1} = \frac{1}{\pi}\int_0^{\infty}\frac{\sqrt{s}(A_{\lambda}(x)-sI)^{-1}}{s+z^2}\,ds.$$

Also, to treat the problem (1)-(2) we will use other hypotheses related to the resolvents of operators $K_{\lambda}(x)$ and their first and second derivatives. For example, we need the assumptions :
For all $z\in S_{\theta_1+\pi/2,r_1}$, the mapping $x\mapsto(K_{\lambda}(x)-zI)^{-1}$, defined on $[0,\infty)$, is in $C^2([0,\infty),L(X))$ and

there exist $C > 0$, $\rho \in (1/2, 1]$ and $\alpha \in (0,1)$ such that for all $z \in S_{\theta_1+\pi/2, r_1}$ and all $x, s \in [0, +\infty)$,

$$\|\frac{\partial}{\partial x}(K_\lambda(x) - zI)^{-1}\|_{L(X)} \leq \frac{C}{|z|^\rho}, \tag{6}$$

$$\|\frac{\partial}{\partial x}(K_\lambda(x) - zI)^{-1} - \frac{\partial}{\partial s}(K_\lambda(s) - zI)^{-1}\|_{L(X)} \leq \frac{C|x-s|^\alpha}{|z|^\rho} \tag{7}$$

with $\alpha + \rho - 1 > 0$

$$\|\frac{\partial^2}{\partial x^2}(K_\lambda(x) - zI)^{-1}\|_{L(X)} \leq C|z|^{1-\rho}, \tag{8}$$

$$\|\frac{d^2}{dx^2}(K_\lambda(x))^{-1} - \frac{d^2}{ds^2}(K_\lambda(s))^{-1}\|_{L(X)} \leq C|x-s|^\alpha. \tag{9}$$

By using the method based on the variation of constant and the Dunford's functional calculus, the solution of problem (1)-(2) can be written as :

$$\begin{aligned} u(x) \quad = \quad & e^{xK_\lambda(x)}\varphi - \frac{1}{2}\int_0^{+\infty} e^{(x+s)K_\lambda(x)}(K_\lambda(x))^{-1}g^*(s)ds \\ & + \frac{1}{2}\int_0^x e^{(x-s)K_\lambda(x)}(K_\lambda(x))^{-1}g^*(s)ds \\ & + \frac{1}{2}\int_x^{+\infty} e^{(s-x)K_\lambda(x)}(K_\lambda(x))^{-1}g^*(s)ds \end{aligned} \tag{10}$$

This is the main result on the existence, the uniqueness and the optimal regularity of the solution :

Theorem 1 *Let $\varphi \in D(A(0))$, $f \in C_\infty^\theta([0, +\infty[; X)$, $0 < \theta < 1$. Then, under our hypotheses, there exists $\lambda^* > 0$ such that for all $\lambda \geq \lambda^*$, the unique classical solution u of problem (1)-(2) satisfies :*

$$u''(.),\ B(.)u'(.),\ A(.)u(.) \in C^\beta([0,1]; X),\ \beta = min(\theta, \eta + \nu - 1),$$

if and only if

$$(-A(0))\varphi + f(0) + \frac{d^2}{dx^2}(\lambda I - A(x))^{-1/2}_{|x=0}(\lambda I - A(0))^{1/2}\varphi \in D_{A(0)}(\theta/2; +\infty).$$

Proof : See [1, 3, 4, 5].

3. CONCLUSION

In this work, by using semigroups theory, the fractional powers of linear operators, the Dunford's functional calculus and the interpolation spaces, we obtained interesting results on the existence, the uniqueness and the maximal regularity of the classical solution of Problem (1)-(2). Moreover, we established necessary and sufficient conditions of compatibility to obtain the solution.

4. REFERENCES

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