SECOND-ORDER DIFFERENTIAL INCLUSION WITH SUM OF TWO PERTURBATIONS

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ABSTRACT

The present paper studies the m-points boundary value problem in a separable Banach space E for the second order perturbed differential inclusion of the form

$$\ddot{u}(t)+\gamma\dot{u}(t)\in F\big(t,u(t),\dot{u}(t)\big)+H\big(t,u(t),\dot{u}(t)\big),\quad \text{a.e.on } [0,1].$$

the existence of solutions is obtained under the assumption that F is an unbounded-valued multifunction and satisfies a pseudo-Lipschitz property and H is lower semi-continuous satisfying $H(t,x,y)\subset\Gamma(t)$, where the multifunction $\Gamma:[0,1]\rightrightarrows E$ is integrably bounded.

1. INTRODUCTION

Second order differential inclusions with bounded perturbations have been studied by several authors. There are many excellent works on the two or three point boundary problems. See for example [3, 4, 11, 13].
Existence of solutions for the second order differential inclusions associated to Lipschitzian multifunctions right-hand sides appeared in a series of works, we can cite [1, 5, 8].
Existence of solutions for the second order differential inclusion of the form $\ddot{u}(t)\in F\big(t,u(t),\dot{u}(t)\big)$ with three-point boundary conditions, where F is a convex compact valued multifunction, Lebesgue-measurable on $[0.1]$, and upper semi-continuous on $E\times E$, under the assumption that $F(t,x,y)\subset\Gamma(t)$ in the case where Γ is integrably bounded, has been studied in [3]. The same differential inclusion where F is unbounded-valued and satisfies a Lipschitz property, has been studied in [1] and pseudo-Lipschitz property, has been studied in [5].
Differential inclusion with sum of two perturbations of the form $\ddot{u}(t)\in F\big(t,u(t),\dot{u}(t)\big)+H\big(t,u(t),\dot{u}(t)\big)$ with three-point boundary conditions, has been studied in [4] where F is a convex compact valued multifunction, Lebesgue-measurable on $[0.1]$, and upper semi-continuous on $E\times E$, H be a multifunction with closed values, Lebesgue-measurable and lower semi-continuous on $E\times E$, under the assumption that $F(t,x,y)\subset\Gamma_1(t)$, $H(t,x,y)\subset\Gamma_2(t)$, in the case where Γ_1,Γ_2 are integrably bounded.
The same differential inclusion with Γ_1,Γ_2 are uniformly Pettis integrable has been stadied in [2].
The aim of our paper is to provide new existence results for problems of m-points conditions associated with differential inclusions.
After the introduction, and the Preliminaries, in section 3, we present the existence of $\mathbf{W}_E^{2,1}([0,1])$-solution for differential inclusion of m-points boundary value $(m>3)$ with a positive coefficient γ in a separable Banach space of the form

$$(\mathscr{P}_{F,H})\begin{cases}\ddot{u}(t)+\gamma\dot{u}(t)\in F\big(t,u(t),\dot{u}(t)\big)+H\big(t,u(t),\dot{u}(t)\big), & \text{a.e. } t\in[0,1]\\ u(0)=0;\quad u(1)=\sum\limits_{i=1}^{m-2}\alpha_i u(\eta_i), & \end{cases}$$

where $F : [0,1] \times E \times E \rightrightarrows E$ is a closed valued multifunction, measurable on $[0,1] \times E \times E$ and satisfies a pseudo-Lipschitz property, that is,

$$v \in F(t,x,y) \Rightarrow d(v, F(t,x',y')) \leq (k_1(t) + \beta_1 \parallel v \parallel) \parallel x - x' \parallel + (k_2(t) + \beta_2 \parallel v \parallel) \parallel y - y' \parallel .$$

and $H : [0,1] \times E \times E \rightrightarrows E$ is another multifunction, with nonempty compact values, lower semi-continuous on $E \times E$ and measurable on $[0,1] \times E \times E$, furthermore $H(t,x,y) \subset \Gamma(t)$, for all $(t,x,y) \in [0.1] \times E \times E$, where Γ is an integrably bounded multifunction, that is, the scalar function
$t \mapsto | \Gamma(t) | := \sup\{\|x\|, x \in \Gamma(t)\}$ is Lebesgue-integrable on $[0.1]$.
The ideas of the proof of our main result are inspired from [5] and [4].
To study this problem, we need first to provide an appropriated Green function associated with the data m and γ in order to establish the existence and uniqueness of a $\mathbf{C}_E^2([0,1])$ (resp. $\mathbf{W}_E^{2,1}([0,1])$) solution for the following ordinary differential equation

$$\begin{cases} \ddot{u}(t) + \gamma \dot{u}(t) = f(t) \;\; t \in [0,1] \\ u(0) = 0; \quad u(1) = \sum\limits_{i=1}^{m-2} \alpha_i u(\eta_i), \end{cases}$$

where f belongs to $\mathbf{C}_E([0,1])$ (resp. $\mathbf{L}_E^1([0,1])$).
In particular, we show some topological properties of solutions set of the differential inclusion with m-points boundary conditions of the form

$$\begin{cases} \ddot{u}(t) + \gamma \dot{u}(t) \in \Gamma(t) \quad \text{a.e. } t \in [0,1] \\ u(0) = 0; \quad u(1) = \sum\limits_{i=1}^{m-2} \alpha_i u(\eta_i), \end{cases}$$

2. MAIN RESULTS

We consider the following assumption.
(A) Let $\gamma > 0$, $m > 3$ be an integer number, $0 < \eta_1 < \eta_2 < ... < \eta_{m-2} < 1$
and $\alpha_i \in \mathbb{R}$ $(i = 1,2,...,m-2)$ satisfying the condition

$$\sum_{i=1}^{m-2} \alpha_i - 1 + \exp(-\gamma) - \sum_{i=1}^{m-2} \alpha_i \exp(-\gamma \eta_i) \neq 0.$$

Proposition 1 *Let (A) holds and let $f \in \mathbf{C}_E([0,1])$ (resp. $f \in \mathbf{L}_E^1([0,1])$. Then the m-points boundary problem*

$$\begin{cases} \ddot{u}(t) + \gamma \dot{u}(t) = f(t), \;\; t \in [0,1] \\ u(0) = 0; \quad u(1) = \sum\limits_{i=1}^{m-2} \alpha_i u(\eta_i) \end{cases}$$

has a unique $\mathbf{C}_E^2([0,1])$-solution (resp. $\mathbf{W}_E^{2,1}([0,1])$-solution) defined by

$$u_f(t) = \int_0^1 G(t,s) f(s) ds, \quad \forall t \in [0,1].$$

The following result is linked to some topological properties of solutions set of the differential inclusion with m-points boundary conditions in Banach spaces. It is the m-points boundary version of the result given by Azzam, Castaing, Thibault [3].

Theorem 2 *Let the assumption (A) holds. Let E be a separable Banach space and let $\Gamma : [0,1] \rightrightarrows E$ be a measurable and integrably bounded multifunction with convex compact values. Then, the $\mathbf{W}_E^{2,1}([0,1])$-solutions set $\mathbf{X}_\Gamma$ of the differential inclusion*

$$\begin{cases} \ddot{u}(t)+\gamma\dot{u}(t) \in \Gamma(t) \quad a.e.\ t \in [0,1] \\ u(0)=0; \quad u(1)=\sum_{i=1}^{m-2}\alpha_i u(\eta_i), \end{cases}$$

is convex compact in the Banach space $\mathbf{C}_E^1([0,1])$ endowed with the norm $\|\ .\ \|_{C^1}$. Furthermore, $\mathbf{X}_\Gamma$ is characterized by

$$\mathbf{X}_\Gamma = \{u_f : [0,1] \to E : u_f(t) = \int_0^1 G(t,s)f(s)ds, \forall t \in [0,1]; f \in \mathbf{S}_\Gamma^1\}$$

where $\mathbf{S}_\Gamma^1$ is the set of all integrable selections of Γ.

$$\mathbf{S}_\Gamma^1 = \{f \in \mathbf{L}_E^1([0,1]), f(t) \in \Gamma(t),\ \forall t \in [0,1]\}.$$

Now we are able to give and prove the existence of $\mathbf{W}_E^{2,1}([0,1])$-solutions for the second order differential inclusion $(\mathscr{P}_{F,H})$.

Theorem 3 *Let the assumption (A) holds. Let E be a separable Banach space and let $F : [0,1] \times E \times E \rightrightarrows E$ be a nonempty closed valued multifunction. Let $g \in \mathbf{L}_E^1([0,1])$ and let $u_g : [0,1] \to E$ be the mapping defined by*

$$u_g(t) = \int_0^1 G(t,s)g(s)ds, \quad \forall t \in [0,1].$$

Assume that for a fixed $\rho \in]0,+\infty]$ and for

$$X_\rho = \{(t,x,y) \in [0,1] \times E \times E : \|x-u_g(t)\| < \rho; \|y-\dot{u}_g(t)\| < \rho\},$$

F is $\mathscr{L}([0,1]) \bigotimes \mathscr{B}(E) \bigotimes \mathscr{B}(E)$-measurable on X_ρ, and there are $\beta_1, \beta_2 \geq 0$ and $k_1(.), k_2(.) \in \mathbf{L}_{\mathbb{R}_+}^1([0,1])$, such that for every $v \in F(t,x,y)$ one has

$$d(v, F(t,x',y')) \leq (k_1(t)+\beta_1 \parallel v \parallel) \parallel x-x' \parallel + (k_2(t)+\beta_2 \parallel v \parallel) \parallel y-y' \parallel . \tag{1}$$

Furthermore, assume that the function $t \mapsto d\big(0, F(t,0,0)\big)$ is integrable.
Let $H : [0,1] \times E \times E \rightrightarrows E$ be another multifunction with nonempty compact values, lower semi-continuous on $E \times E$ and $\mathscr{L}([0,1]) \bigotimes \mathscr{B}(E) \bigotimes \mathscr{B}(E)$-measurable.
Assume that, there exists a convex $\|.\|$-compact valued, and measurable multifunction $\Gamma : [0,1] \to E$ which is integrably bounded, such that $H(t,x,y) \subset \Gamma(t)$ for all $(t,x,y) \in [0,1] \times E \times E$.
Then the differential inclusion $(\mathscr{P}_{F,H})$ has at least a solution $u \in W_E^{2,1}([0,1])$.

Corollary 4 *Let the assumption (A) holds. Let E be a separable Banach space and let $F : [0,1] \times E \times E \to E$ be a nonempty closed valued multifunction such that*
F is $\mathscr{L}([0,1]) \bigotimes \mathscr{B}(E) \bigotimes \mathscr{B}(E)$-measurable, there are $\beta_1, \beta_2 \geq 0$ and $k_1(.), k_2(.) \in \mathbf{L}_{\mathbb{R}_+}^1([0,1])$, such that for every $v \in F(t,x,y)$ one has

$$d(v, F(t,x',y')) \leq (k_1(t)+\beta_1 \parallel v \parallel) \parallel x-x' \parallel + (k_2(t)+\beta_2 \parallel v \parallel) \parallel y-y' \parallel;$$

and the function $t \mapsto d\big(0, F(t,0,0)\big)$ is integrable.
Let $H : [0,1] \times E \times E \rightrightarrows E$ be another multifunction with nonempty compact values, lower semi-continuous on $E \times E$ and $\mathscr{L}([0,1]) \bigotimes \mathscr{B}(E) \bigotimes \mathscr{B}(E)$-measurable.
Assume that, there exists a convex $\|.\|$-compact valued, and measurable multifunction $\Gamma : [0,1] \to E$ which is integrably bounded, such that $H(t,x,y) \subset \Gamma(t)$ for all $(t,x,y) \in [0,1] \times E \times E$.
Then the differential inclusion $(\mathscr{P}_{F,H})$ has at least a solution $u \in W_E^{2,1}([0,1])$.

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