MULTIPLE SOLUTIONS FOR NONHOMOGENEOUS ELLIPTIC EQUARTIONS INVOLVING CRITICAL CAFFARLLI-KOHN-NIRENBERG EXPONRENT

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ABSTRACT

In this work, we consider a nonhomogeneous singular elliptic equation involving a critical Caffarelli-Kohn-Nirenberg exponent with singular terms. By using the Nehari manifold we establish the existence of two solutions.

1. INTRODUCTION

We study the existence of multiple solutions to the nonhomogeneous problem

$$(\mathscr{P})\begin{cases} -div(\frac{|\nabla u|^{p-2}}{|x|^{pa}}\nabla u)-\mu\frac{|u|^{p-2}}{|x|^{p(a+1)}}u=\frac{|u|^{p^*-2}}{|x|^{p^*b}}u+f(x) \text{ in } \Omega, & \\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ $(N\geq 3)$ containing 0 in its interior, $1<p<N,\ -\infty<a<(N-p)/p,\ a\leq b<a+1,\ p^*=pN/(N-pa-p+pb)$ is the critical Caffarelli-Kohn-Nirenberg exponent, $-\infty<\mu<\overline{\mu},\overline{\mu}:=[(N-pa-p)/p]^p$ and f is a given measurable function different than 0.
We shall work with the space $W_\mu^{1,p}:=W_\mu^{1,p}(\Omega)$ for $-\infty<\mu<\overline{\mu}$ endowed with the norm

$$\|u\|_\mu^p:=\int_\Omega\left(\frac{|\nabla u|^p}{|x|^{pa}}-\mu\frac{|u|^p}{|x|^{pa+p}}\right)dx.$$

which is equivalent to the norm $\|.\|_0$.

To state our result, let set for $u\in W_\mu^{1,p}$ and $f\in W_\mu^*$(the dual of $W_\mu^{1,p}$)

$$\|u\|^{p^*}:=(\int_\Omega\frac{|u|^{p^*}}{|x|^{p^*b}}dx)^{\frac{1}{p^*}},$$

$$I_f(u):=\int_\Omega fu\,dx,\qquad S_\mu:=\inf_{\|u\|^{p^*}=1}\|u\|_\mu^p,$$

$$\gamma_f:=\inf_{\|u\|^{p^*}=1}\left\{(p^*-p)\left[\frac{1}{p^*-1}\|u\|_\mu^p\right]^{\frac{p^*-1}{p^*-p}}-I_f(u)\right\}$$

and

$$D:=\left\{g\in W_\mu^*,\ g\neq 0;\ \gamma_g>0\right\}.$$

Note $D \neq \varnothing$, notice that if $f \in L^p(\Omega)$ then

$$\int_\Omega |f|^p \, dx < (p^* - p)^p \left[\frac{1}{(p^*-1)}\right]^{\frac{p(p^*-1)}{p^*-p}} S_\mu^{p^*/(p^*-p)}.$$

which implies that $f \in D$.

.The purpose of this paper is to investigate the existence of a ground state solution for the problem (P) by a "smallness" condition on f.

2. THE MAIN RESULT

The main result is concluded as the following theorem, which is new for the singular case when $p \neq 2$.

Theorem 1 *Let $-\infty < a < (N-p)/p$, $a \leq b < a+1$, and $-\infty \leq \mu < \bar{\mu}$. Assume that $f \in D$, then (P) has two solutions*

To prove the theorem we give some results which will be used to establish the existence of solutions, we know by [6] that $S_\mu > 0$ and is attained when $\Omega = \mathbb{R}^N$.

Since $f \in W_\mu^*$ then the Euler-Lagrange functional I associated to the problem (P) is given by

$$I(u) = \frac{1}{p}\|u\|_\mu^p - \frac{1}{p^*}\|u\|_{p^*}^{p^*} - I_f(u) \quad \text{for all } u \in W_\mu^{1,p},$$

it's clear that $I \in C^1(W_\mu^{1,p}, \mathbb{R})$ and satisfies

$$\langle I'(u), v\rangle = (\int_\Omega \frac{|\nabla u|^{p-2}}{|x|^{2a}} \nabla u \nabla v - \mu \frac{|u|^{p-2}}{|x|^{p(a+1)}} uv - \frac{|u|^{p^*-2}}{|x|^{2_* b}} uv - fv) dx$$

for all u, $v \in W_\mu^{1,p}$.

Hince, weak solution of (P) are critical points of the fonctional I.

3. THE NEHARI MANIFOLD

We denote the Nehari manifold by

$$\mathcal{N} = \left\{ u \in W_\mu^{1,p} / \{0\}, \langle I'(u), u\rangle = 0 \right\}.$$

It is easy to see that $u \in \mathcal{N}$ if and only if

$$J(u) = \|u\|_\mu^p - \|u\|_{p^*}^{p^*} - I_f(u) = 0.$$

Lemma 2 *The fonction I is coercive and bounded from below in $\mathcal{N}$.*

The Nehari manifold $\mathcal{N}$ is closely linked to the behavior of the form $\Phi_u(t) : t \to I(tu)$, which for $t > 0$ is defined by

$$\Phi_u(t) = \frac{t^p}{p}\|u\|_\mu^p - \frac{t^{p^*}}{p^*}\|u\|_{p^*}^{p^*} - tI_f(u).$$

Lemma 3 *Let $u \in W_\mu^{1,p}$, then $tu \in \mathcal{N}$ if and only if $\Phi_u'(t) = 0$.*

The elements in $\mathcal{N}$ correspond to stationary points of the maps Φ_u.
We note that
$$\Phi_u'(t) = t^{p-1}\|u\|_\mu^p - t^{\,p^*-1}\|u\|_{p^*}^{p^*} - I_f(u).$$
and
$$\Phi_u^{"}(t) = (p-1)t^{p-2}\|u\|_\mu^p - (p^*-1)t^{\,p^*-2}\|u\|_{p^*}^{p^*}\quad.$$
By Lemma 2 we have $u \in \mathcal{N}$ if and only if $\Phi_u'(1) = 0$.Hince
$$\Phi_u^{"}(1) = (p-1)\|u\|_\mu^p - (p^*-1)\|u\|_{p^*}^{p^*}\quad.$$

Then it is naturalto split $\mathcal{N}$ into three subsets corresponding to local minima, local maxima, and poind of inflexion, i.e,
$$\mathcal{N}^+ = \left\{u \in \mathcal{N} : \Phi_u^{"}(1) > 0\right\},\ \mathcal{N}^- = \left\{u \in \mathcal{N} : \Phi_u^{"}(1) < 0\right\},$$
and
$$\mathcal{N}^0 = \left\{u \in \mathcal{N} : \Phi_u^{"}(1) = 0\right\}.$$
First, we prove that $\Phi_u^{"}(1) \neq 0$ for all $u \in \mathcal{N}/\{0\}$.

Lemma 4 *Assume that* $f \in D$ *. Then* $\mathcal{N}^0 = \varnothing$.

Define for all $u \in W_\mu^{1,p}/\{0\}$
$$t_u^{\max} := \left[\|u\|_\mu^p (p-1)/(p^*-1)\|u\|_{p^*}^{p^*}\right]^{\frac{1}{p^*-p}}.$$

Lemma 5 *Assume that* $f \in D$ *. Then for any* $u \in W_\mu^{1,p}/\{0\}$*, there exists a unique positive value* t_u^+ *such that*
$$t_u^+ > t_u^{\max},\ t_u^+ u \in \mathcal{N}^- \text{ and } I\left(t_u^+ u\right) = \max_{t \geq t_u^{\max}} I(tu).$$
Moreover, if $I_f(u) > 0$*, then there exists a unique positive value* t_u^- *such that*
$$0 < t_u^- < t_u^{\max},\ t_u^- u \in \mathcal{N}^+ \text{ and } I\left(t_u^- u\right) = \inf_{0 \leq t \leq t_u^{\max}} I(tu).$$

By the previous lemma we know that $\mathcal{N}^+$ and $\mathcal{N}^-$ are not empty, so we can define
$$\theta^+ := \inf_{u \in \mathcal{N}^+} I(u) \text{ and } \theta^- := \inf_{u \in \mathcal{N}^-} I(u).$$

Lemma 6 *Assume that* $f \in D$ *. Then for any* $u \in \mathcal{N}^\pm$*, there exist* $\varepsilon > 0$ *and a differentiable function* $\zeta = \zeta(v)$, $v \in W_\mu^{1,p}$, $\|v\|_\mu < \varepsilon$*, such that* $\xi(0) = 1$, $\zeta(v)(u-v) \in \mathcal{N}^\pm$ *and*
$$\left(\zeta'(0), v\right) = \frac{\int_\Omega \left[p\left(\frac{|\nabla u|^{p-2}\nabla u\,\nabla v}{|x|^{pa}} - \mu\frac{u^{p-2}uv}{|x|^{p(a+1)}}\right) - p^*\frac{|u|^{p^*-2}uv}{|x|^{p^*b}} - fv\right]dx}{(p-1)\|u\|_\mu^p - (p^*-1)\|u\|_{p^*}^{p^*}}.$$

Lemma 7 *Let* $f \in D$*, then there exist* $\theta_0^+ < 0$ *and* $\theta_0^- > 0$*such that* $\theta^+ \leq \theta_0^+$ *and.* $\theta^- \geqslant -\theta_0^-$.

Lemma 8 *Assume that* $f \in D$*. Then, there exists a minimizing sequence* (u_n) *such that*
$$I(u_n) \longrightarrow \theta^+ \text{ and } I'(u_n) \to 0 \text{ in } W_\mu^*.$$

4. CONCLUSION

By the Nehari manifold we proved the existence of two solutions, such that $u_1 \in \mathscr{N}^-$ and $u_2 \in \mathscr{N}^+$,where $\mathscr{N}^- \cap \mathscr{N}^+ = \emptyset$ then $u_1 \neq u_2$.

5. REFERENCES

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