PROBABILITY TAIL FOR LINEARLY NEGATIVE QUADRANT DEPENDENT RANDOM VARIABLES OF PARTIAL SUMS AND APPLICATION TO LINEAR MODEL

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ABSTRACT

In this paper, we establish a new concentration inequality and complete convergence of weighted sums for arrays of rowwise linearly negative quadrant dependent (LNQD, in short) random variables and obtain a result dealing with complete convergence of first-order autoregressive processes with identically distributed LNQD innovations.
key words : Complete convergence ; Linearly Negative Quadrant Dependent Random Variables ; Autoregressive Process.
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1. INTRODUCTION

The concept of complete convergence of a sequence of random variables was introduced by Hsu and Robbins [3] as follows. A sequence $\{X_n, n \geq 1\}$ of random variables converges completely to the constant C if

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n - C| > \varepsilon) < \infty \text{ for all } \varepsilon > 0.$$

By the Borel-Cantelli lemma, this implies $X_n \to C$ almost surely (a.s.), and the converse implication is true if the $\{X_n, n \geq 1\}$ are independent. Hsu and Robbins [4] proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Erdös [1] proved the converse. This result has been generalized and extended in several directions and carefully studied by many authors (see, Gut[3], Kuczmaszewska and Szynal[10], Ghosal and Chandra[2], Hu et al.[5, 6]). Complete convergence for sequence of random variables plays a central role in the area of limit theorems in probability theory and mathematical statistics. Conditions of independence and identical distribution of random variables are basic in historic results due to Bernoulli, Borel and Kolmogorov. Since then, serious attempts have been made to relax these strong conditions. For example, independence has been relaxed to pairwise independence or pairwise negative quadrant dependence or, even replaced by conditions of dependence such as mixing or martingale. In particular, many authors showed that many results could be obtained by replacing i.i.d. condition by uniformly bounded condition. We recall that an array $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$

of random variables is said to be stochastically dominated by a nonnegative random variable X (write $\{X_{ni} \prec X\}$) if there exists a constant $C > 0$ such that

$$\mathbb{P}(|X_{ni}| > t) \leq C\mathbb{P}(X > t) \; \forall \; t > 0 \; , \; n \geq 1, \; 1 \leq i \leq n. \tag{1}$$

The main purpose of this paper, is to discuss the complete convergence for sums of row-wise linearly negative quadrant dependent (LNQD, in short) random variables under suitable conditions, since independent and identically random variables are a special case of linearly negative quadrant dependent random variables. The exponential inequality plays an important role in various proofs of limit theorems. In particular, it provides a measure of the complete convergence for partial sums. The exponential inequality for negatively associated (NA, in short) random variables has been studied by many authors; see, for example, [5,7,10], and so forth. The main purpose of this work is to extend the exponential inequality for NA random variables to the case of LNQD random variables. In addition, we obtain the complete convergence for $S_n = \sum_{i=1}^{n} X_i$, which improves on the corresponding ones of [4-6]. Lehmann [11] introduced a simple and natural definition of negative dependence : A sequence $\{\zeta_i, 1 \leq i \leq n\}$ of random variables is said to be pairwise negative quadrant dependent (pairwise NQD) if for any real $\varepsilon_i, \varepsilon_j$ and $i \neq j, \mathbb{P}(\zeta_i > \varepsilon_i, \zeta_j > \varepsilon_j) \leq \mathbb{P}(\zeta_i > \varepsilon_i)\mathbb{P}(\zeta_j > \varepsilon_j)$: Much stronger concept than NQD was considered by Joag-Dev and Proschan [7] : A sequence $\{\zeta_i, 1 \leq i \leq n\}$ is said to be negatively associated(NA) if for any disjoint subsets, $A, B \subset \{1, 2, ..., n\}$ and any real coordinatewise increasing functions f on $\mathbb{R}^A$ and g on $\mathbb{R}^B$, $Cov(f(\zeta_i, i \in A), g(\zeta_i, i \in B)) \leq 0$. Instead of negative association, Newman [12] noticed that his method of proof yielding the central limit theorem for negatively associated sequence requires only that positive linear combinations of the random variables are NQD, i.e., the random variables are linearly negative quadrant dependent (LNQD). This notion of negative dependence was formulated by Newman [12] as follows : $\{\zeta_i, i \in \mathbb{N}\}$ is a sequence of LNQD random variables if for any disjoint subsets A, B of $\mathbb{N}$ and positive r_i, the random vector $(\sum_{i \in A} r_i\zeta_i; \sum_{i \in B} r_i\zeta_i)$ is NQD. Negatively associated sequences are LNQD and LNQD sequences are not necessarily NA, as it can be seen from examples in Newman [12] or Joag-Dev [7].

We note also that negative association and its weaker concepts are of considerable use in probability and statistics (cf. Joag-Dev and Proschan [7], Newman [12] and the references there in). Newmann [12] was first to establish a central limit theorem for LNQD random variables, Kim et al.[9] derived a general central limit theorem for weighted sum of LNQD random variables.

Firstly, we will recall the definitions of negatively associated, negative quadrant dependent and linearly negative quadrant dependent sequence.

Definition 1 *[11] Two random variables ζ_1 and ζ_2 are said to be negative quadrant dependent (NQD, in short) if for any $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$,*

$$\mathbb{P}(\zeta_1 < \varepsilon_1, \zeta_2 < \varepsilon_2) \leq \mathbb{P}(\zeta_1 < \varepsilon_1)\mathbb{P}(\zeta_2 < \varepsilon_2). \tag{2}$$

A sequence $\{\zeta_n, n \geq 1\}$ of random variables is said to be pairwise NQD if ζ_i and ζ_j are NQD for all $i, j \in \mathbb{N}^+$ and $i \neq j$.

Definition 2 *[12] A sequence $\{\zeta_n, n \geq 1\}$ of random variables is said to be linearly negative quadrant dependent (LNQD, in short) if for any disjoint subsets $A, B \subset$ and positive $r_j^{'}s$,*

$$\sum_{k \in A} r_k\zeta_k \textit{ and } \sum_{j \in B} r_j\zeta_j \textit{ are NQD.}$$

Remark 1 *It is easily seen that if $\{\zeta_n, n \geq 1\}$ is a sequence of LNQD random variables, then $\{a\zeta_n + b, n \geq 1\}$ is still a sequence of LNQD random variables, where a and b are real numbers.*

Lemma 1 *[11] Let random variables X and Y be NQD. Then*
(i) $\mathbb{E}(XY) \leq \mathbb{E}(X)\mathbb{E}(Y)$ *;*
(ii) $\mathbb{P}(X > x, Y > y) \leq \mathbb{P}(X > x)\mathbb{P}(Y > y)$ *;*
(iii) If f and g are both nondecreasing (or both nonincreasing) functions, then $f(X)$ and $g(Y)$ are NQD.

Lemma 2 *Let $\{X_n, n \geq 1\}$ be a sequence of LNQD random variables and $t > 0$, then for each $n \geq 1$,*

$$\mathbb{E}\left[\prod_{i=1}^{n} \exp(tX_i)\right] \leq \prod_{i=1}^{n} \mathbb{E}(\exp(tX_i)) \tag{3}$$

Proof. For $t > 0$, it is easy to see that tX_i and $t\sum_{j=i+1}^{n} X_j$ are NQD by the definition of LNQD, which implies that $\exp(tX_i)$ and $\exp(t\sum_{j=i+1}^{n} X_j)$ are also NQD for $i = 1, 2, \ldots, n-1$ by Lemma**??**(iii). It follows from Lemma**??**(i) and induction that

$$\begin{aligned}
\mathbb{E}\left[\prod_{i=1}^{n} \exp(tX_i)\right] &= \mathbb{E}\left[\exp(tX_1)\exp\left(\sum_{i=2}^{n} tX_i\right)\right] \\
&\leq \mathbb{E}\left[\exp(tX_1)\right]\mathbb{E}\left[\exp\left(\sum_{i=2}^{n} tX_i\right)\right] \\
&= \mathbb{E}\left[\exp(tX_1)\right]\mathbb{E}\left[\exp(tX_2)\exp\left(\sum_{i=3}^{n} tX_i\right)\right] \\
&\leq \mathbb{E}\left[\exp(tX_1)\right]\mathbb{E}\left[\exp(tX_2)\right]\mathbb{E}\left[\exp\left(\sum_{i=3}^{n} tX_i\right)\right] \\
&\leq \prod_{i=1}^{n} \mathbb{E}(\exp(tX_i)).
\end{aligned}$$

This completes the proof of the lemma.
Throughout the paper, let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be a sequence of random variables defined on a fixed probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Denote $S_n = \sum_{i=1}^{n} X_{ni}$ and $B_n = \sum_{i=1}^{n} \mathbb{E}(X_{ni}^2)$ for each $1 \leq i \leq n$ and $n \geq 1$.

2. MAIN RESULTS

Lemma 3 *Let $\alpha > 0$ constants and $0 < \beta \leq \dfrac{\alpha^2}{e^{\alpha} - 1 - \alpha}$. Then*

$$\exp(x) - 1 - x \leq \frac{x^2}{\beta} \tag{4}$$

for all $0 \leq x \leq \alpha$

Proof.Consider the function

$$\Psi(x, \beta) = \ln(1 + x + \frac{x^2}{\beta}) - x.$$

We need to prove that $\Psi(x,\beta) \geq 0$ for all
$0 < \beta \leq \dfrac{\alpha^2}{e^{\alpha}-1-\alpha}$ and $0 \leq x \leq \alpha$.
Take the derivative

$$\frac{\partial \Psi(x,\beta)}{\partial x} = -\frac{x(x-(2-\beta))}{\beta(1+x+\frac{x^2}{\beta})}.$$

Hence, Ψ is increasing in x on the interval $(0, 2-\beta)$ and decreasing on the interval $(2-\beta, \alpha)$. Note that $\Psi(0,\beta) = 0$ and $\Psi(\alpha,\beta) \geq 0$ since $0 < \beta \leq \dfrac{\alpha^2}{e^{\alpha}-1-\alpha}$

Let

$$X_{1,ni} = -a_n 1_{\{X_{ni}<-a_n\}} + X_{ni} 1_{\{|X_{ni}|\leq a_n\}} + a_n 1_{\{X_{ni}>a_n\}},$$

$$X_{2,ni} = (X_{ni}-a_n) 1_{\{X_{ni}>a_n\}}, \tag{5}$$

$$X_{3,ni} = (X_{ni}+a_n) 1_{\{X_{ni}<-a_n\}}. \tag{6}$$

Here, and in the sequel, 1_A denotes the indicator function of the A set in the braces, that is, it takes value 1 or 0 according to whether or not the sample point belongs to the set.
It is easy to check that $X_{1,ni} + X_{2,ni} + X_{3,ni} = X_{ni}$ for $1 \leq i \leq n, n \geq 1$ and $X_{1,n1}, X_{1,n2}, \ldots, X_{1,nn}$ are bounded by a_n for each fixed $n \geq 1$.
If $\{X_{ni}, n \geq 1\}$ are LNQD random variables, then $\{X_{p,ni}, 1 \leq i \leq n\},\ p = 1,2,3$ are also LNQD random variables for each fixed $n \geq 1$.

Theorem 4 *Let $\{X_n, n \geq 1\}$ be a sequence of LNQD random variables with $\mathbb{E}X_i = 0$. If there exists a positive constants α, λ such that $0 \leq X_i \leq \frac{\alpha}{\lambda}, i \geq 1$ then for any $\lambda \geq 0$,*

$$\mathbb{E}\exp\left\{\lambda \sum_{i=1}^{n} X_i\right\} \leq \exp\left\{\frac{\lambda^2}{\beta} \sum_{i=1}^{n} \mathbb{E}X_i^2\right\} \tag{7}$$

Proof. By using Lemma 3 and Lemma 2, we can see that

$$\mathbb{E}\exp\left\{\lambda \sum_{i=1}^{n} X_i\right\} \leq \prod_{i=1}^{n} \mathbb{E}e^{\lambda X_i} \leq \exp\left\{\frac{\lambda^2}{\beta} \sum_{i=1}^{n} \mathbb{E}X_i^2\right\} \tag{8}$$

Corollary 5 *Let $\{X_n, n \geq 1\}$ be a sequence of LNQD random variables. If there exists a positive constants α, λ such that*
$0 \leq X_i \leq \frac{\alpha}{\lambda}, i \geq 1$ then for any $\lambda \geq 0$,

$$\mathbb{E}\exp\left\{\lambda \sum_{i=1}^{n} (X_i - \mathbb{E}X_i)\right\} \leq \exp\left\{\frac{\lambda^2}{\beta} \sum_{i=1}^{n} \mathbb{E}X_i^2\right\} \tag{9}$$

Proof. It is easily seen that $\{X_n - \mathbb{E}X_n, n \geq 1\}$ is a sequence of LNQD random variables with $\mathbb{E}(X_i - \mathbb{E}X_i) = 0$. By Theorem 4, we have

$$\mathbb{E}\exp\left\{\lambda \sum_{i=1}^{n} (X_i - \mathbb{E}X_i)\right\} \leq \exp\left\{\frac{\lambda^2}{\beta} \sum_{i=1}^{n} \mathbb{E}(X_i - \mathbb{E}X_i)^2\right\}$$

$$\leq \exp\left\{\frac{\lambda^2}{\beta} \sum_{i=1}^{n} \mathbb{E}X_i^2\right\}$$

Theorem 6 *Let $\{X_n, n \geq 1\}$ be a sequence of LNQD random variables such that $0 \leq X_i \leq \frac{\alpha}{\lambda}, i \geq 1$ where α and λ a positive constants. Then for any $\varepsilon \geq 0$ we have*

$$\mathbb{P}\left(\sum_{i=1}^{n}(X_i - \mathbb{E}X_i) \geq \varepsilon\right) \leq \exp\left\{-\frac{\varepsilon^2\beta}{4B_n}\right\} \tag{10}$$

Proof. By Markov's inequality and lemma 3, we have that for any $\lambda > 0$.

$$\begin{aligned} \mathbb{P}\left(\sum_{i=1}^{n}(X_i - \mathbb{E}X_i) \geq \varepsilon\right) &\leq e^{-\lambda\varepsilon}\mathbb{E}\exp\left\{\sum_{i=1}^{n}(X_i - \mathbb{E}X_i)\right\} \\ &\leq \exp\left\{-\lambda\varepsilon + \frac{\lambda^2}{\beta}B_n^2\right\} \end{aligned}$$

Taking $\lambda = \frac{\varepsilon\beta}{2B_n}$, we can obtain 10

$$\mathbb{P}\left(\sum_{i=1}^{n}(X_i - \mathbb{E}X_i) \leq -\varepsilon\right) \leq \mathbb{P}\left(\sum_{i=1}^{n}(-X_i - \mathbb{E}(-X_i)\right) \geq \varepsilon)$$

$$\leq \exp\left\{-\frac{\varepsilon^2\beta}{4B_n}\right\} \tag{11}$$

since $\{-X_n, n \leq 1\}$ is a sequence of LNQD random variables.

Theorem 7 *Let $\{X_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of rowwise LNQD random variables with $\mathbb{E}X_{ni} = 0$, and $\{a_n, n \geq 1\}$ a sequence of positive constants. Suppose that*

(i) $\sum_{n=1}^{\infty}\exp\{-\frac{\beta\varepsilon^2}{4a_n}\} < \infty$ *for some* $0 < \beta \leq \dfrac{\alpha^2}{e^{\alpha} - 1 - \alpha}$ *and* $|X_{ni}| \leq \alpha$.

(ii) $\sum_{i=1}^{n}\mathbb{E}(X_{ni}^2) = O(a_n)$,

Then $\sum_{i=1}^{n}X_{ni}$ *converges completely to zero.*

Proof. From the inequality $\exp(x) \leq 1 + x + \dfrac{x^2}{\beta}$ for all $0 \leq x \leq \alpha$ and $0 < \beta \leq \dfrac{\alpha^2}{e^{\alpha} - 1 - \alpha}$ (see lemma 3), we have by (i) that for any $\lambda > 0$

$$\begin{aligned} \mathbb{E}\exp(\lambda X_{ni}) &\leq \mathbb{E}\left\{1 + \lambda X_{ni} + \frac{1}{2\beta}\lambda^2|X_{ni}|^2\right\} \\ &= 1 + \frac{1}{2\beta}\lambda^2\mathbb{E}|X_{ni}|^2 \\ &\leq \exp\left\{\frac{1}{2\beta}\lambda^2\mathbb{E}|X_{ni}|^2\right\} \end{aligned}$$

The second inequality follows by the fact that $1 + t \leq e^t$ for all real number t. It follows by

Markov's inequality, Lemma 2, and (i) that for any $\lambda > 0$,

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^{n} X_{ni} > \varepsilon\right) &\leq e^{-\lambda\varepsilon}\mathbb{E}\exp(\lambda\sum_{i=1}^{n} X_{ni}) \\ &\leq e^{-\lambda\varepsilon}\prod_{i=1}^{n}\mathbb{E}\exp(\lambda X_{ni}) \\ &\leq e^{-\lambda\varepsilon}\exp\left\{\frac{1}{2\beta}\lambda^2\sum_{i=1}^{n}\mathbb{E}|X_{ni}|^2\right\} \\ &\leq e^{-\lambda\varepsilon}\exp\left\{\frac{1}{2\beta}\lambda^2 O(a_n)\right\} \\ &= \exp\left\{-\lambda\varepsilon + \frac{1}{2\beta}\lambda^2 O(a_n)\right\}.\end{aligned}$$

Choosing $\lambda = \dfrac{\varepsilon\beta}{2O(a_n)}$, we have that for all large n,

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^{n} X_{ni} > \varepsilon\right) &\leq \exp\left\{-\varepsilon^2\frac{\beta}{2}O(a_n) + \frac{\varepsilon^2\beta^2 O(a_n)}{4\beta(O(a_n))^2}\right\} \\ &= \exp\left\{-\varepsilon^2\frac{\beta}{4O(a_n)}\right\} \\ &\leq \exp\left\{-\varepsilon^2\frac{\beta}{4a_n}\right\}.\end{aligned}$$

Thus by (i)

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\sum_{i=1}^{n} X_{ni} > \varepsilon\right) < \infty. \tag{12}$$

Since $\{-X_{ni}, 1 \leq i \leq n, n \geq 1\}$ is still an array of row-wise LNQD random variables, we can replace X_{ni} by $-X_{ni}$ from the above statement. That is,

$$\sum_{n=1}^{\infty}\mathbb{P}\left(\sum_{i=1}^{n} X_{ni} < -\varepsilon\right) < \infty. \tag{13}$$

The result follows by (12) and (13).
Now we state and prove our main result.

3. APPLICATIONS OF THE RESULTS TO AR(1) MODEL

The basic object of this section is applying the results to first-order autoregressive processes(AR(1)).

3.1. The AR(1) model

We consider an autoregressive time series of first order AR(1) defined by

$$X_{n+1} = \theta X_n + \zeta_{n+1},\ n = 1, 2, ..., \tag{14}$$

where $\{\zeta_n, n \geq 0\}$ is a sequence of identically distributed LNQD random variables with $\zeta_0 = X_0 = 0$, $0 < \mathbb{E}\zeta_k^4 < \infty$, $k = 1, 2, ...$ and where θ is a parameter with $|\theta| < 1$. Here, we can rewrite X_{n+1} in (14) as follows :

$$X_{n+1} = \theta^{n+1} X_0 + \theta^n \zeta_1 + \theta^{n-1} \zeta_2 + ... + \zeta_{n+1}. \tag{15}$$

The coefficient θ is fitted least squares, giving the estimator

$$\widehat{\theta}_n = \frac{\sum_{j=1}^{n} X_j X_{j-1}}{\sum_{j=1}^{n} X_{j-1}^2} \tag{16}$$

It immediately follows from (14) and (16) that

$$\widehat{\theta}_n - \theta = \frac{\sum_{j=1}^{n} \zeta_j X_{j-1}}{\sum_{j=1}^{n} X_{j-1}^2} \tag{17}$$

Theorem 8 *Let the conditions of theorem (3) be satisfied then for any* $\frac{(\mathbb{E}X_1^2)^{\frac{1}{2}}}{R} < \zeta$ *positive, with take* $B_n = \sum_{i=1}^{n} \mathbb{E}X_i^2$, *we have*

$$\mathbb{P}(\sqrt{n}|\widehat{\theta}_n - \theta| > R) \leq \left[\exp\{-\frac{n^2(R^2\zeta^2 - \mathbb{E}X_1)}{4B_n}\} + \exp\{-\frac{\mathbb{E}X_{j-1}^2 - n\zeta^2}{4\mathbb{E}X_{j-1}^4}\} \right] \tag{18}$$

where $\mathbb{E}X_j^2 \leq \infty$ *and* $\mathbb{E}X_j^4 \leq \infty$.

Proof.
Firstly, we notice that :

$$\widehat{\theta}_n - \theta = \frac{\sum_{j=1}^{n} \zeta_j X_{j-1}}{\sum_{j=1}^{n} X_{j-1}^2}$$

It follows that

$$\mathbb{P}(\sqrt{n}|\widehat{\theta}_n - \theta| > R) = \mathbb{P}\left(\left| \frac{1/\sqrt{n} \sum_{j=1}^{n} \zeta_j X_{j-1}}{1/n \sum_{j=1}^{n} X_{j-1}^2} \right| > R \right)$$

By virtue of the probability properties and Hölder's inequality, we have for any $\widetilde{\varepsilon}$ positive

$$\begin{aligned} \mathbb{P}(\sqrt{n}|\widehat{\theta}_n - \theta| > R) &\leq \mathbb{P}(1/n \sum_{j=1}^{n} X_j \geq R^2\zeta^2) + \mathbb{P}(1/n^2 \sum_{j=1}^{n} X_{j-1}^2 \leq R^2) \\ &= \mathbb{P}(\sum_{j=1}^{n} X_j \geq (R^2\zeta^2)n) + \mathbb{P}(\sum_{j=1}^{n} X_{j-1}^2 \leq n^2\zeta^2) \\ &= I_{1n} + I_{2n}. \end{aligned}$$

Next we estimate I_{1n} and I_{2n}.

Corollary 9 *The sequence* $(\widehat{\theta}_n)_{n\in\mathbb{N}}$ *defined in (16) completely converges to the parameter* θ *of the first-order autoregressive process.*

4. CONCLUSIONS

The exponential probability inequalities have been important tools in probability and statistics. In this paper, we prove an new exponential inequalities for the distributions of sums of linearly negative quadrant dependent (LNQD, in short) random variables, and obtain a result dealing with complete convergence of first-order autoregressive processes with identically distributed (LNQD) innovations.

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