SOME WEAK INVARIANCE RESULTS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT

We investigate weak invariance for a graph of a given set–valued map with respect to a given fractional differential inclusion. A tangency condition is defined and extends in a natural way those used to get weak invariance for ordinary differential inclusions.

1. INTRODUCTION

We consider a fractional differential inclusion

$$D_a^{\alpha} y \in F(t, y(t)) \tag{1}$$

where $F : [a,b) \times \mathbb{R}^n \rightsquigarrow \mathbb{R}^n$ is a given set–valued map and D_a^{α} stands for the Caputo fractional derivative of order α with $\alpha \in (0,1)$. Let $G : [a,b) \rightsquigarrow \mathbb{R}^n$ and denote by $\mathscr{K}$ the graph of G, that is

$$\mathscr{K} = \{(t,x); x \in G(t)\}.$$

For various reasons, solutions of (1) need to satisfy constraints which are described, in the theoretical approach, by a graph $\mathscr{K}$.

Weak invariance theorems characterize the connection between the dynamics (in the study problem (1)) and a given constraint, to guarantee the existence for any initial state in the constraint, of at least one solution starting from that initial state and such that its graph lies in the constraint for some time.

To be more precise, we say that $\mathscr{K}$ is (exact) weak invariant with respect to (1), if for any $(t_0,x) \in \mathscr{K}$ there exist $T > t_0$ with $[t_0,T] \subset [a,b)$ and a solution $y : [t_0,T] \to \mathbb{R}^n$ of (1) with $y(t_0) = x$ such that $(t, y(t)) \in \mathscr{K}$, for every $t \in [t_0,T]$. Of course the notion of solution of (1) must be properly defined later on.

Traditionally, for ordinary differential inclusion criteria of weak invariance have been given in terms of tangency condition. We recall the pioneering contribution of Nagumo [15] in 1942 who considered the particular case of a differential equation ($\alpha = 1$ and $F \equiv f$ is single valued), i.e.,

$$y'(t) = f(t, y(t)), \tag{2}$$

with $f : [a,b) \times K \to \mathbb{R}^n$ is continuous and K is closed or merely locally closed.

Definition 1 *A subset $K \subset X$ is locally closed if for every $x \in K$ there exists $\rho > 0$ such that $K \cap B(x,\rho)$ is closed.*

We notice that Nagumo used the term "*rechtz zulassing*" in German, whose English translation is *right admissible* to design viable, and proved that a necessary and sufficient condition in order that $I \times K$ be weak invariant with respect to (2) in case when $f(\cdot,\cdot)$ is continuous and K is locally closed is the tangency condition :

(TC) for each $(t_0,x) \in \mathscr{K}$,

$$\liminf_{h\to 0^+} \frac{1}{h} \text{dist}(x+hf(t_0,x);K) = 0.$$

Here, $\text{dist}(\eta,K)$ stands for the distance from $\eta \in \mathbb{R}^n$ to the set K, i.e.

$$\text{dist}(\eta,K) = \inf_{u\in K} \|\eta - u\|.$$

Since the tangency condition (TC) is obviously satisfied at each point $(t_0,x) \in I \times K$ with x is an interior point of K, we deduce that, whenever the function $f(\cdot,\cdot)$ is continuous and the subset K is open (which is of course locally closed), then $I \times K$ is weak invriant with respect to (2). But this is nothing else that the celebrated Peano's local existence theorem [17], proved by G. Peano in 1890.

Theorem 1 *(Peano [17]) Let $f : I \times K \to \mathbb{R}^n$ be continuous, and $K \subset \mathbb{R}^n$ be open. Then for every $(t_0,x) \in I \times K$, there exist $T > t_0$ with $[t_0,T] \subset I$ and a solution* [1] *$y : [t_0,T] \to \mathbb{R}^n$ of* (2) *on $[\tau,T]$ satisfying $y(t_0) = x$.*

The situation is not easy for fractional differential inclusion. The case when $\alpha \in (0,1)$ has been treated in many papers. Unfortunately, up to our knowledge, none of these works characterizes explicitly the weak invariance of a set with respect to (1). Precisely, we may show some results in that context. Indeed, many papers deals with a generalization of the classical proof of Nagumo–Brezis theorem in the classical weak invariance theory. However, it seems that unfortunately the proof of their results is not correct. The reason, in our opinion, is that the tangency condition used is badly formulated.

In this paper we give some new results in the context of weak invariance for fractional differential inclusion and discuss some ideas in that direction.

2. PRELIMINARIES

2.1. (Exact) and and near weak invariance for a graph wit respect to fractional differential inclusion

In the following we assume that $y : [a,b] \to \mathbb{R}^n$ is absolutely continuous that is

$$\forall (t,s) \subset [a,b], y(t) - y(s) = \int_s^t y'(\theta) d\theta.$$

Definition 2 *The Caputo derivative of y of order α is defined by :*

$$D_a^\alpha y = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-s)^{-\alpha} y'(s) ds; s \in (a,b).$$

Here and thereafter Γ stands for the Euler gamma function.

1. By a solution of (2) on $[t_0,T] \subset I$, we mean a continuously differentiable function $y : [t_0,T] \to \mathbb{R}^n$ satisfying (2) for every $t \in [t_0,T]$. We note here that Nagumo used also the same notion of solution.

Definition 3 *By a solution of* (1) *on* $[t_0,T] \subset [a,b)$ *we mean an absolutely continuous function* $y : [t_0,T] \to \mathbb{R}^n$ *for which there exists a mesurable selection* f_y *satisfying* $f_y(t) \in F(t,y(t))$*, a.e.*$t \in [t_0,T]$ *and for all* $t \in [t_0,T]$ *one has :*

$$y(t) = y(t_0) + \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} (t-s)^{\alpha-1} f_y(s) ds.$$

Definition 4 1. *We say that* $\mathscr{K}$ *is weak invariant with respect to* (1) *if for each* $(t_0,x) \in \mathscr{K}$*, there exist* $T > t_0$ *with* $[t_0,T] \subset [a,b)$ *and a solution* $y : [t_0,T] \to \mathbb{R}^n$ *of* (1) *satisfying*

$$\forall t \in [t_0,T], y(t) \in G(t).$$

2. *The graph* $\mathscr{K}$ *is said to be near weak invariant with respect to* (1) *if for each* $(t_0,x) \in \mathscr{K}$*, there exists* $T > t_0$ *with* $[t_0,T] \subset [a,b)$ *and for each* $\varepsilon > 0$ *there exists a solution* $y_\varepsilon : [t_0,T] \to \mathbb{R}^n$ *to* (1) *satisfying :*

$$\forall t \in [t_0,T], \operatorname{dist}(y_\varepsilon(t); G(t)) \leq \varepsilon.$$

2.2. Tangency condition

Let $E : [a,b) \rightsquigarrow \mathbb{R}^n$ be a given set–valued map and let $(t_0,x) \in \mathscr{G}$. We denote by $\mathscr{S}_{[t_0,t_0+h]}E(\cdot)$ the set of all integrable selections of $E(\cdot)$ defined on $[t_0,t_0+h]$ for some $h > 0$ with $[t_0,t_0+h] \subset [a,b)$.

It is well known that if $E(\cdot)$ is measurable with nonempty and closed values then by Kuratowski-Ryll Nardzewski theorem [1, Theorem 8.1.3], the set–valued map $E(\cdot)$ has at least one measurable selection. Further, if $E(\cdot)$ is integrally bounded, i.e., for some $l \in L^1(I;\mathbb{R}_+)$ we have $E(t) \subset l(t)\mathbb{B}$ for a.e. $t \in I$, then each measurable selection of $E(\cdot)$ is integrable thanks to Lebesgue theorem.

In the following we introduce the notion of tangent set–values map which generalizes the one given in [11].

Definition 5 *We say that* $E(\cdot)$ *is quasi–tangent to* $\mathscr{G}$ *at* (t_0,x) *if*

$$\liminf_{h \to 0^+} \frac{1}{h} \operatorname{dist}\left(\left\{ y(t_0+h); y(t_0) = x, \text{ and } f_y \in \mathscr{S}_{[t_0,t_0+h]}E(\cdot) \right\}; G(t_0+h) \right) = 0.$$

Next we give some characterizations of the above tangency condition that prove useful in the sequel.

Theorem 2 *Let* $(t_0,x) \in \mathscr{K}$*. The following conditions are equivalent :*

(i) $E(\cdot)$ *is quasi–tangent to* $\mathscr{G}$ *at* (t_0,x)*;*

(ii) *there exist two sequences,* $(h_n)_n$ *in* $\mathbb{R}_+$ *with* $h_n \downarrow 0$ *and* $(f_n)_n$ *such that* $f_n = f_{y_n}$*,* $y_n(t_0) = x$ *and* $f_n \in S_{[t_0,t_0+h_n]}E(\cdot)$ *for each* $n \in \mathbb{N}$*, satisfying*

$$\liminf_{n \to +\infty} \frac{1}{h_n} \operatorname{dist}\bigg(y_n(t_0+h_n); K(t_0+h_n) \bigg) = 0;$$

(iii) *there exist three sequences,* $(h_n)_n$ *in* $\mathbb{R}_+$ *with* $h_n \downarrow 0$*,* $(f_n)_n$ *such that* $f_n = f_{y_n}$*,* $y_n(t_0) = x$*,* $f_n \in S_{[t_0,t_0+h_n]}E(\cdot)$ *for each* $n \in \mathbb{N}$ *and* $(p_n)_n$ *in* X *with* $\lim_{n\to+\infty} p_n = 0$*, satisfying*

$$y_n(t_0+h_n) + h_n p_n \in K(t_0+h_n)$$

for every $n \in \mathbb{N}$*. In addition, if* X *is a separable Banach space then (i)∼(iii) are equivalent to*

(iv) *the set–valued function* $\overline{co}E(\cdot)$ *is quasi–tangent to* $\mathscr{G}$ *at* (t_0,x)*.*

3. THE MAIN RESULT

The main result of the paper is given below. We prove that a necessary condition of weak invariance of the graph $\mathscr{K}$ with respect to (1) is an appropriate tangency condition. More precisely we have.

Theorem 3 *The graph $\mathscr{K}$ is weak invariant with respect to* (1) *if $F(\cdot,x)$ is quasi–tangent to $\mathscr{G}$ at (t_0,x) for every $(t_0,x)\in\mathscr{G}$.*

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