SOLVABILITY FOR A CLASS OF NONLINEAR FRACTIONAL RELAXATION DIFFERENTIAL EQUATIONS

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ABSTRACT

This paper is concerned with a class of nonlinear fractional relaxation differential equations. The existence and uniqueness results are obtained by applying some fixed point theorems. Finally, we provide an example to illustrate the obtained results.

1. INTRODUCTION

The concept of fractional calculus is a generalization of the ordinary differentiation and integration to arbitrary non integer order. Fractional differential equations with and without delay arise from a variety of applications including in various fields of science and engineering such as applied sciences, practical problems concerning mechanics, the engineering technique fields, economy, control systems, physics, chemistry, biology, medicine, atomic energy, information theory, harmonic oscillator, nonlinear oscillations, conservative systems, stability and instability of geodesic on Riemannian manifolds, dynamics in Hamiltonian systems, etc. In particular, problems concerning qualitative analysis of linear and nonlinear fractional differential equations with and without delay have received the attention of many authors, see [1, 2, 3, 6] and the references therein.
Inspired and motivated by the studied works in [1, 2, 3, 6], in this paper, we study the existence and uniqueness of solution for the following nonlinear fractional relaxation differential equation

$$\begin{cases} {}^{C}D^{\alpha}\left(u(t)-g(t,u(t))\right)+\omega u(t)=f(t,u(t)),\ \omega>0, t\in(0,T] \\ u(0)=u_0\in\mathbb{R}, \end{cases} \tag{1}$$

where ${}^{C}D^{\alpha}$ is the Caputo fractional derivative of order $\alpha\in(0,1]$, $f:[0,T]\times\mathbb{R}\to\mathbb{R}$ and $g:[0,T]\times\mathbb{R}\to\mathbb{R}$ are nonlinear continuous functions. To show the existence and uniqueness of solutions, we transform (1) into an integral equation and then use the Banach and Krasnoselskii fixed point theorems. Finally, we provide an example to illustrate our obtained results.

The rest of this paper is organized as follows. Some definitions from fractional calculus theory are recalled in Section 2. In Section 3, we prove the existence and uniqueness of solutions for (1). Finally, in Section 4, we give an example to illustrate the usefulness of our main results.

2. PRELIMINARIES

In this section we present some basic definitions, notations and results of fractional calculus which are used throughout this paper.

Let $T>0$, $J=[0,T]$. By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|u\|=\sup\{u(t):t\in J\}.$$

Definition 1 ([5]) *The fractional integral of order $\alpha > 0$ of a function $u : J \to \mathbb{R}$ is given by*

$$I^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\,ds,$$

provided the right side is pointwise defined on J.

Definition 2 ([5]) *The Caputo fractional derivative of order $\alpha > 0$ of function $u : J \to \mathbb{R}$ is given by*

$${}^{C}D^{\alpha}u(t) = I^{n-\alpha}D^{(n)}u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s)\,ds,$$

where $n = [\alpha]+1$, provided the right side is pointwise defined on J.

Definition 3 ([2]) *The two-parameter function of the Mittag-Leffler type is defined by the series expansion*

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+\beta)},\ \alpha > 0,\ \beta \in \mathbb{C},\ z \in \mathbb{C}.$$

For $\beta = 1$, we obtain the Mittag-Leffler function in one parameter

$$E_{\alpha}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)},\ \alpha > 0,\ z \in \mathbb{C}.$$

Lemma 1 ([2]) *For $0 < \alpha \leq 1$, the Mittag-Leffler type function $E_{\alpha,\alpha}(-\omega t^{\alpha})$ satisfies*

$$0 \leq E_{\alpha,\alpha}(-\omega t^{\alpha}) \leq \frac{1}{\Gamma(\alpha)},\ t \in [0,\infty),\ \omega \geq 0,$$

and

$$\lim_{t\to 0^+} E_{\alpha,\alpha}(-\omega t^{\alpha}) = E_{\alpha,\alpha}(0) = \frac{1}{\Gamma(\alpha)}.$$

Lemma 2 ([4]) *For $t \in [0,\infty)$ and $0 < \alpha \leq 1$, the one-parameter Mittag-Leffler function $E_{\alpha,1}(-t^{\alpha})$ is decreasing function of t and it is bounder from above by 1, that is*

$$E_{\alpha,1}(-\omega t^{\alpha}) \leq 1.$$

Furthermore, it is to be noted that

$$\lim_{t\to\infty} E_{\alpha,1}(-\omega t^{\alpha}) = 0.$$

Lastly in this section, we state the fixed point theorems which enable us to prove the existence and uniqueness of a solution of (1).

Theorem 3 (Banach's fixed point theorem [7]) *Let Ω be a non-empty closed subset of a Banach space $(S, \|.\|)$, then any contraction mapping Φ of Ω into itself has a unique fixed point.*

Theorem 4 (Krasnoselskii's fixed point theorem [7]) *Let Ω be a non-empty bounded closed convex subset of a Banach space $(S, \|.\|)$. Suppose that F_1 and F_2 map Ω into S such that*
(i) $F_1u + F_2v \in \Omega$ for all $u, v \in \Omega$,
(ii) F_1 is continuous and compact,
(iii) F_2 is a contraction.
Then there is a $u \in \Omega$ with $F_1u + F_2u = u$.

3. EXISTENCE AND UNIQUENESS

Let us start by defining what we mean by a solution of the problem (1).

Definition 4 *A function $u \in C^1(J,\mathbb{R})$ is said to be a solution of problem (1) if u satisfies ${}^{C}D^{\alpha}(u(t)-g(t,u(t)))+\omega u(t)=f(t,u(t))$ for any $t\in J$ and $u(0)=u_0$.*

For the existence of solutions for the problem (1), we need the following auxiliary lemma.

Lemma 5 *Let $u\in C(J,\mathbb{R})$ and u' exists, then u is a solution of the initial value problem (1) if and only if it is a solution of the integral equation*

$$\begin{aligned} u(t) &= (u_0-g(t,u_0))E_{\alpha,1}(-\omega t^{\alpha})+g(t,u(t)) \\ &+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\omega(t-s)^{\alpha})\left[f(s,u(s))-\omega g(s,u(s))\right]ds. \end{aligned} \tag{2}$$

Proof. It is easy to prove by the Laplace transform. ■

In the following subsections we prove existence, as well as existence and uniqueness results for the problem (1) by using a variety of fixed point theorems.

The following assumptions will be used in our main results

(H1) There exists a constant $L_f\in\mathbb{R}^+$ such that

$$|f(t,u)-f(t,v)|\le L_f|u-v|,$$

for $t\in J$, $u,v\in\mathbb{R}$.

(H2) There exists a constant $L_g\in(0,1)$ such that

$$|g(t,u)-g(t,v)|\le L_g|u-v|,$$

for $t\in J$, $u,v\in\mathbb{R}$.

3.1. Existence and uniqueness results via Banach's fixed point theorem

Theorem 6 *Assume that the assumptions (H1)–(H2) are satisfied. If*

$$L_g+\frac{T^{\alpha}}{\Gamma(\alpha+1)}\left(L_f+L_g\omega\right)<1. \tag{3}$$

Then there exists a unique solution for the problem (1) on J.

Proof. We define the operator $\Phi: C(J,\mathbb{R})\to C(J,\mathbb{R})$ by

$$\begin{aligned} (\Phi u)(t) &= (u_0-g(t,u_0))E_{\alpha,1}(-\omega t^{\alpha})+g(t,u(t)) \\ &+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\omega(t-s)^{\alpha})\left[f(s,u(s))-\omega g(s,u(s))\right]ds. \end{aligned}$$

Clearly, the fixed points of operator Φ are solutions of problem (1). For any $u,v\in C([0,T],\mathbb{R})$ and $t\in J$, we have

$$\begin{aligned} &|(\Phi u)(t)-(\Phi v)(t)| \\ &\le |g(t,u(t))-g(t,v(t))| \\ &+\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\omega(t-s)^{\alpha})|f(s,u(s))-f(s,v(s))|ds \\ &+\omega\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\omega(t-s)^{\alpha})|g(s,u(s))-g(s,v(s))|ds. \end{aligned}$$

By (H1) and (H2), we have

$$\begin{aligned} &\left|(\Phi u)(t)-(\Phi v)(t)\right| \\ &\leq L_g \|u-v\| + \frac{T^\alpha L_f}{\Gamma(\alpha+1)} \|u-v\| + \frac{T^\alpha L_g \omega}{\Gamma(\alpha+1)} \|u-v\|, \end{aligned}$$

thus

$$\|\Phi u - \Phi v\| \leq \left(L_g + \frac{T^\alpha}{\Gamma(\alpha+1)} \left(L_f + L_g \omega \right) \right) \|u-v\|.$$

From (3), Φ is a contraction. As a consequence of Banach's fixed point theorem, we get that Φ has a unique fixed point which is a unique solution of the problem (1) on J. ■

3.2. Existence results via Krasnoselskii's fixed point theorem

Theorem 7 *Assume (H2) and the following hypotheses*
(H3) There exist $p_1 \in C\left(J, \mathbb{R}^+\right)$ such that

$$|f(t,u)| \leq p_1(t),$$

for $t \in J$ and each $u \in \mathbb{R}$.
(H4) There exist $p_2 \in C\left(J, \mathbb{R}^+\right)$ such that

$$|g(t,u)| \leq p_2(t),$$

for $t \in J$ and each $u \in \mathbb{R}$.
Then the problem (1) has at least one solution in Ω.

Proof. Let us fix

$$\rho \geq |u_0| + q + p_2^* + \frac{T^\alpha}{\Gamma(\alpha+1)} \left(p_1^* + \omega p_2^*\right),$$

where $p_1^* = \sup_{t\in J} p_1(t)$, $p_2^* = \sup_{t\in J} p_2(t)$ and $q = \sup_{t\in J} |g(t,u_0)|$. Consider the non-empty closed convex subset

$$\Omega = \left\{u \in C(J,\mathbb{R}),\ \|u\| \leq \rho\right\},$$

and define two operators F_1 and F_2 on Ω, as follows

$$(F_1 u)(t) = \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\omega(t-s)^\alpha) \left[f(s,u(s)) - \omega g(s,u(s))\right] ds,$$

and

$$(F_2 u)(t) = (u_0 - g(t,u_0)) E_{\alpha,1}(-\omega t^\alpha) + g(t,u(t)).$$

We shall use the Krasnoselskii fixed point theorem to prove there exists at least one fixed point of the operator $F_1 + F_2$ in Ω. The proof will be given in the following steps.
Step 1. We prove that $F_1 u + F_2 v \in \Omega$ for all $u, v \in \Omega$.
Step 2. We prove that F_1 is compact and continuous.
Step 3. We prove that $F_2 : \Omega \to C(J,\mathbb{R})$ is a contraction mapping.
Clearly, all the hypotheses of the Krasnoselskii fixed point theorem are satisfied. Hence, there exists a fixed point $u \in \Omega$ such that $u = F_1 u + F_2 u$ which is a solution of the problem (1). ■

Example 1 *We consider the fractional initial value problem*

$$\begin{cases} {}^C D^{\frac{1}{2}} \left(u(t) - \frac{1}{4} t \sin(u(t))\right) + \frac{1}{2} u(t) = \frac{1}{(\exp(t)+4)(|u(t)|+1)},\ t \in J = [0,1], \\ u(0) = 1, \end{cases} \tag{4}$$

where $T=1$, $u_0=1$, $\alpha=\omega=\frac{1}{2}$, $g(t,u)=\frac{1}{4}t\sin(u)$ *and* $f(t,u)=\frac{1}{(\exp(t)+4)(|u|+1)}$. *For each* $u,v\in\mathbb{R}$ *and* $t\in J$, *we have*

$$\begin{aligned}|f(t,u)-f(t,v)| &= \left|\frac{1}{(\exp(t)+4)(|u|+1)}-\frac{1}{(\exp(t)+4)(|v|+1)}\right| \\ &\leq \frac{|u-v|}{(\exp(t)+4)(1+|u|)(1+|v|)} \\ &\leq \frac{1}{5}|u-v|,\end{aligned}$$

and

$$|g(t,u)-g(t,v)|\leq\frac{1}{4}|u-v|.$$

Hence, assumptions (H1) and (H2) are satisfied with $L_f=\frac{1}{5}$ *and* $L_g=\frac{1}{4}$. *The condition*

$$L_g+\frac{T^\alpha}{\Gamma(\alpha+1)}\left(L_g\omega+L_f\right)\simeq 0.62<1,$$

is satisfied. It follows from Theorem 6 that the problem (4) has a unique solution on J.

4. CONCLUSIONS

We can conclude that the main results of this work have been successfully achieved, that is, through the Banach contraction principle and the Krasnoselskii fixed point theorem , we scrutinized the existence and uniqueness of solutions for a class of nonlinear fractional relaxation differential equations.

5. REFERENCES

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