ASYMPTOTIC BEHAVIOUR OF SOLUTIONS OF NONLOCAL ELLIPTIC PROBLEMS

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ABSTRACT

Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \geq 1$. This work aims to provide the asymptotic behavior of solutions of nonlocal elliptic problems associated with a sequence of measurable, bounded and uniformly elliptic matrices on Ω using the notion of H-convergence.

Key words. Asymptotic behavior ; nonlocal elliptic problems ; H-convergence.

1. INTRODUCTION

For a rael parameter $\varepsilon > 0$ and a.e. $x = (x_1,...,x_n) \in \Omega$, let $a^\varepsilon(x) = (a^\varepsilon_{ij}(x))_{ij}$ be an $n \times n$ matrix functions satisfying for some positive constants $0 < \lambda \leq \Lambda$,

$$\forall \xi \in \mathbb{R}^n, \quad \text{a.e. } x \in \Omega : \quad \lambda|\xi|^2 \leq a^\varepsilon(x)\xi \cdot \xi \leq \Lambda|\xi|^2, \tag{1}$$

$$a^\varepsilon(x) \text{ is a symmetric matrix.} \tag{2}$$

Let C_Ω be a Poincaré constant for $H^1_0(\Omega)$. Let f be a function defined on $\Omega \times \mathbb{R}$ such that $x \mapsto f(x,t)$ is measurable for all $t \in \mathbb{R}$, $f(.,0) \in L^2(\Omega)$, $f(.,0) \neq 0$ a.e. in Ω and

$$\forall t,s \in \mathbb{R}, \text{ a.e. } x \in \Omega, \quad |f(x,t) - f(x,s)| \leq b(x)|t-s|^p \tag{3}$$

with $p \in (0,1)$, $b \in L^{\frac{2}{1-p}}(\Omega)$ and let M be a continuous function on $(0,+\infty)$ that satisfies

$$M(t) \geq t^\beta \quad \forall t \in (0,+\infty), \tag{4}$$

where β is a real number satisfying one of following assumptions :

$$\beta \in \Big(\frac{p-1}{2}, 0\Big) \cup \Big(0, \frac{1}{2}\Big] \text{ with } q = 2; \tag{5}$$

$$\beta \in \Big(\frac{p-1}{q}, 0\Big) \text{ with } q \in [1,2); \tag{6}$$

$$\beta = \frac{p-1}{q} \text{ with } q \in [1,2] \text{ and } \frac{|\Omega|^{-\frac{2-q}{2}\beta}}{\Lambda_1} C^2_\Omega |b|_{\frac{2}{1-p}} < 1. \tag{7}$$

Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that $x \mapsto g(x,t)$ is measurable for all $t \in \mathbb{R}$, $t \mapsto g(x,t)$ is continuous on $\mathbb{R}$ for a.e. $x \in \Omega$, $g(.,0) \neq 0$ a.e. in Ω and

$$\exists h \in L^2(\Omega), \forall t \in \mathbb{R}, \text{ a.e. } x \in \Omega, \quad |g(x,t)| \leq h(x), \tag{8}$$

and let N be a function defined and continuous on $(0,+\infty)$ such that

$$N(t) \geq t^{\gamma} \quad \forall t \in (0,+\infty), \tag{9}$$

where γ is a real number that satisfies one of the following cases

$$\gamma \in \Big(-\frac{1}{2},0\Big) \cup \Big(0,\frac{1}{2}\Big] \text{ with } q=2; \tag{10}$$

$$\gamma \in \Big(-\frac{1}{q},0\Big) \text{ with } q \in [1,2). \tag{11}$$

We now consider the following weak formulations of nonlocal elliptic problems for the matrix $a^{\varepsilon}(x)$ ([6]),

$$(\mathbf{P_1^{\varepsilon}}) \quad \begin{cases} \text{Find } u^{\varepsilon} \in H_0^1(\Omega) \text{ such that :} \\ \\ M(|u^{\varepsilon}|_q^q)\displaystyle\int_{\Omega} a^{\varepsilon}(x)\nabla u^{\varepsilon}\cdot\nabla\xi\,dx = \int_{\Omega} f(x,u^{\varepsilon})\xi\,dx \\ \forall \xi \in H_0^1(\Omega) \end{cases}$$

and

$$(\mathbf{P_2^{\varepsilon}}) \quad \begin{cases} \text{Find } u^{\varepsilon} \in H_0^1(\Omega) \text{ such that :} \\ \\ N(|u^{\varepsilon}|_q^q)\displaystyle\int_{\Omega} a^{\varepsilon}(x)\nabla u^{\varepsilon}\cdot\nabla\xi\,dx = \int_{\Omega} g(x,u^{\varepsilon})\xi\,dx \\ \forall \xi \in H_0^1(\Omega). \end{cases}$$

In [6], the author established the existence of a nontrivial solution for (P_1^{ε}) and (P_2^{ε}) according to the two bundles of hypotheses mentioned above by using an approximation method.

By the theory of homogenization (see [2] and [5]), the assumptions (1)-(2) imply that there exists a subsequence, still denoted by ε, and a matrix $a^0(x)$ satisfying (1)-(2) such that for every $f \in H^{-1}(\Omega)$, the sequence v^{ε} of the (unique) solutions of the Dirichlet boundary value problems

$$(E^{\varepsilon}) \begin{cases} -div(a^{\varepsilon}(x)\nabla v^{\varepsilon}) = f & \text{in } \mathscr{D}'(\Omega) \\ v^{\varepsilon} \in H_0^1(\Omega) \end{cases}$$

satisfies

$$v^{\varepsilon} \rightharpoonup v^0 \quad \text{weakly in } H_0^1(\Omega),$$
$$a^{\varepsilon}(x)\nabla v^{\varepsilon} \rightharpoonup a^0(x)\nabla v^0 \quad \text{weakly in } \mathbb{L}^2(\Omega),$$

where v^0 is the unique solution of the problem

$$(E^0) \begin{cases} -div(a^0(x)\nabla v^0) = f & \text{in } \mathscr{D}'(\Omega) \\ v^0 \in H_0^1(\Omega). \end{cases}$$

In this case, we will say that the sequence a^{ε} of matrices H-converges to the matrix a^0, and we will write $a^{\varepsilon} \rightharpoonup^H a^0$.

Under assumptions that $h : \Omega \times \mathbb{R} \to \mathbb{R}$ has a subcritical or critical growth with respect to Sobolev imbedding, the authors in [1] studied the behavior, with respect to the H-convergence of the elliptic matrices $a^{\varepsilon}(x)$, of positive solutions of

$$-div(a^{\varepsilon}(x)\nabla u^{\varepsilon}) = h(x,u^{\varepsilon}), \quad u^{\varepsilon} > 0 \text{ in } \Omega, \quad u^{\varepsilon} = 0 \text{ on } \partial\Omega$$

to a solution u_0 of

$$-div(a^0(x)\nabla u^0) = h(x,u^0), \quad u^0 > 0 \text{ in } \Omega, \quad u^0 = 0 \text{ on } \partial\Omega.$$

In this work, we study the asymptotic behavior of solutions of nonlocal elliptic problems (P_1^ε) and (P_2^ε) as $\varepsilon \to 0$ under weak conditions on the diffusion coefficients M and N using the notion of H-convergence.

2. MAIN RESULTS : ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF (P_1^ε) AND (P_2^ε)

Here our main results.

Theorem 1 *Assume that (3) holds. If M and β satisfy (4) one of* (5), (6), (7)*, respectively, there exists a subsequence ε_k such that $(u_1^{\varepsilon_k})_{k\in\mathbb{N}}$ converges weakly in $H_0^1(\Omega)$ to a nontrivial function u_1^0 solution of the problem :*

$$(\mathbf{P_1^0}) \begin{cases} \textit{Find } u_1^0 \in H_0^1(\Omega) \textit{ such that}: \\ M(|u_1^0|_q^q)\displaystyle\int_\Omega a^0(x)\nabla u_1^0\cdot\nabla\xi\, dx = \int_\Omega f(x,u_1^0)\xi\, dx \\ \forall \xi \in H_0^1(\Omega). \end{cases}$$

Theorem 2 *Under assumptions (8) and that the function N and the real number γ satisfy (9) and one of* (10) *and* (11)*, respectively, there exists a subsequence ε_k a nontrivial function u_2^0 solution of the problem :*

$$(\mathbf{P_2^0}) \begin{cases} \textit{Find } u_2^0 \in H_0^1(\Omega) \textit{ such that}: \\ N(|u_2^0|_q^q)\displaystyle\int_\Omega a^0(x)\nabla u_2^0\cdot\nabla\xi\, dx = \int_\Omega g(x,u_2^0)\xi\, dx \\ \forall \xi \in H_0^1(\Omega) \end{cases}$$

such that $(u_2^{\varepsilon_k})_{k\in\mathbb{N}}$ converges weakly in $H_0^1(\Omega)$ to u_2^0.

Idea of the proof. First, under assumptions on M and f, we prove that $(u_1^\varepsilon)_{\varepsilon>0}$ is bounded in $H_0^1(\Omega)$ that there exists a constant c independently of ε such that

$$\forall \varepsilon > 0: \quad \|u^\varepsilon\| \le c, \tag{12}$$

then, form (12) and the compact immersion $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$, we can extract a subsequence, also noted ε_k, and $u_1^0 \in H_0^1(\Omega)$ such that

$$u_1^{\varepsilon_k} \rightharpoonup u_1^0 \qquad \text{weakly in } H_0^1(\Omega), \tag{13}$$

$$u_1^{\varepsilon_k} \to u_1^0 \qquad \text{strongly in } L^2(\Omega), \tag{14}$$

$$|u_1^{\varepsilon_k}|_q^q \to |u_1^0|_q^q \qquad \text{strongly in } \mathbb{R}. \tag{15}$$

Next, we consider as test functions the solutions $v^{\varepsilon_k} \in H_0^1(\Omega)$ of (E^{ε_k}), $\varepsilon_k \ge 0$ and we assume that $v^0 \in \mathscr{D}(\Omega)$. Then, we have the following lemmas :

Lemma 3 *Set $\eta^{\varepsilon_k} = a^{\varepsilon_k}(x)\nabla u_1^{\varepsilon_k}$. There exists a subsequence still denoted by ε_k and $\eta^0 \in L^2(\Omega)$ such that*

$$\forall \varphi \in \mathscr{D}(\Omega), \quad \lim_{k\to+\infty}\int_\Omega \eta^{\varepsilon_k}.\nabla v^{\varepsilon_k}\varphi\, dx = \int_\Omega \eta^0.\nabla v^0\varphi\, dx.$$

Lemma 4 *We have*

$$\forall \varphi \in \mathscr{D}(\Omega), \quad \lim_{k\to+\infty}\int_\Omega \varphi\, a^{\varepsilon_k}(x)\nabla u_1^{\varepsilon_k}.\nabla v^{\varepsilon_k}\, dx = \int_\Omega \varphi\, a^0(x)\nabla u_1^0.\nabla v^0\, dx.$$

Now, from Lemmas 3 and 4, we deduce that

$$\forall v^0 \in \mathscr{D}(\Omega), \quad \eta^0.\nabla v^0 = a^0(x)\nabla u_1^0.\nabla v^0 \quad \text{a.e. in } \Omega$$

and then

$$\eta^0 = a^0(x)\nabla u_1^0 \quad \text{a.e. in } \Omega. \tag{16}$$

Finally, passing to the limit in $(P_1^{\varepsilon_k})$ as $k \to +\infty$ and using (1), (3), the continuity of M and (13)-(16) to see that u_1^0 is a solution to (P_1^0).

Remark 1 *Similarly, we prove Theorem 2.*

3. CONCLUSIONS

Under boundedness and generalized Lipschitz conditions on the reaction terms and weak assumptions on the diffusion coefficients, we studied the asymptotic behavior of solutions of nonlocal elliptic problems associated with the sequence of matrices $(a^\varepsilon)_{\varepsilon>0}$ as $\varepsilon \to 0$ using the notion of H-convergence.

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