MULTIPLE NONTRIVIAL SOLUTIONS FOR A CLASS OF NONLINEAR ELLIPTIC KIRCHHOFF EQUATIONS

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ABSTRACT

This work is devoted to the existence and multiplicity to a Brezis-Nirenberg type problems involving singular nonlinearitie. The main tool is variational methods, more precisely, by using the Ekeland's variational principle we can find the first critical point with negative level. From the Mountain Pass Theorem we also obtain a critical point whose level is positive.

1. INTRODUCTION

We are concerned with the existence and multiplicity of solutions to the following Kirchhoff problem

$$(\mathscr{P}_\lambda)\begin{cases} -\left(a\|u\|^2_{\alpha,\mu}+b\right)\left(div\left(\frac{\nabla u}{|x|^{2\alpha}}\right)+\mu\frac{|u|}{|x|^{2(\alpha+1)}}\right)=\frac{|u|^{p^*-2}}{|x|^{p^*\beta}}u+\lambda f(x) \ \text{ in } \mathbb{R}^N \\ u\in W^{1,2}_{\alpha,\mu}\left(\mathbb{R}^N\right) \end{cases}$$

where $N\geq 3,\ a>0,\ b\geq 0,\ -\infty<\mu<\overline{\mu}:=\left[\frac{(N-2(\alpha+1))}{2}\right]^2$ and $\lambda>0$, with $0\leq\alpha<(N-2)/2$, $\alpha\leq\beta<\alpha+1$ and $p^*=\frac{2N}{N-2(1+\alpha-\beta)}$ is the critical Caffarelli-Kohn-Nirenberg exponent, f satisfies some conditions and

$$\|u\|^2_{\alpha,\mu}:=\int_{\mathbb{R}^N}\left(\frac{|\nabla u|^2}{|x|^{2\alpha}}-\mu\frac{|u|^2}{|x|^{2(\alpha+1)}}\right)dx.$$

Where $W^{1,2}_{\alpha,\mu}\left(\mathbb{R}^N\right)=W^{1,2}_{\alpha,\mu}$ a Banach space, W^* dual of $W^{1,2}_{\alpha,\mu}$.
We define the energy functional associated to the problem $(\mathscr{P}_\lambda)$ by

$$I_\lambda(u)=\frac{a}{4}\|u\|^4+\frac{b}{2}\|u\|^p-\frac{1}{p^*}\int_{\mathbb{R}^N}\frac{|u|^{p^*}}{|x|^{p^*\beta}}dx-\lambda\int_{\mathbb{R}^N}f(x)u dx,\ \ \forall u\in W^{1,2}_{\alpha,\mu}.$$

The functional I_λ is well defined in $W^{1,2}_{\alpha,\mu}$ and belongs to $C^1\left(W^{1,2}_{\alpha,\mu},\mathbb{R}\right)$ and that we have

$$\left\langle I'_\lambda(u),v\right\rangle=\left(b+a\|u\|^2\right)\int_{\mathbb{R}^N}\nabla u\nabla v-\int_{\mathbb{R}^N}\frac{|u|^{p^*-2}}{|x|^{p^*\beta}}uv+\lambda\int_{\mathbb{R}^N}f(x)v\,dx.$$

for all $v\in W^{1,2}_{\alpha,\mu}$.

Furthermore, every critical point of I_λ is a weak solution of $(\mathscr{P}_\lambda)$. Thus in the following sections we shall prove the existence of a nontrivial critical points of I_λ.

In order to critical point theory we first derive results related to the Palais-Smale compactness condition.

Definition 1 *We call a sequence $(u_n) \in W^{1,2}_{\alpha,\mu}$ a Palais-Smale sequence on $W^{1,2}_{\alpha,\mu}$ if $I(u_n) \to c$ and $\|I'(u_n)\|_{W^*} \to 0$ as $n \to +\infty$.*

Definition 2 *Let $c \in \mathbb{R}$. We say that I satisfies the Palais Smale condition at level c, if any Palais-Smale sequence contains a convergent subsequence in $W^{1,2}_{\alpha,\mu}$.*

Theorem 1 *If $W^{1,2}_{\alpha,\mu}$ is a Banach space and $I \in C^1(W^{1,2}_{\alpha,\mu}, \mathbb{R})$ is bounded from below, then there exists a minimizing sequence (u_n) for I in $W^{1,2}_{\alpha,\mu}$ such that*

$$I(u_n) \to \inf_{W^{1,2}_{\alpha,\mu}} I \text{ and } I'(u_n) \to 0 \text{ in } W^* \text{ as } n \to +\infty.$$

Theorem 2 *Let $I \in C^1(W^{1,2}_{\alpha,\mu}, \mathbb{R})$ satisfies (PS) condition. Assume*

(1) $I(0) = 0$,

(2) there exist two numbers ρ and α such that $I(u) \geq \alpha$ for every $u \in \left\{ u \in W^{1,2}_{\alpha,\mu} : \|u\| = \rho \right\}$,

(3) there exist $v \in V$ such that $I(v) < \alpha$ and $\|v\| \geq \rho$.

Define

$$\Gamma := \left\{ \gamma \in C\left([0,1],\ W^{1,2}_{\alpha,\mu}\right),\ \gamma(0) = 0,\ \gamma(1) = v \right\},$$

and $c := \inf_{\gamma\in\Gamma} \max_{u\in\gamma([0,1])} I(u) \geq \alpha$ is a critical value.

2. THE MAIN RESULT

To state our result, we make the following assumptions.

(Hf) $f \in W^* \backslash \{0\}$, doesn't change its sign or $\int_{\mathbb{R}^N} f(x) V_\varepsilon dx > 0$.

$(H1)$ $3 \leq N \leq 4$, $\beta - \alpha = 1 - \frac{N}{4}$, $0 < a < (S_\mu)^{-2}$, $b > 0$.

$(H2)$ $N = 3$, $\beta - \alpha < \frac{1}{4}$, $a > 0$, $b = 0$.

$(H3)$ $N = 3$, $\beta - \alpha = 0$, $a > 0$, $b > 0$.

The main results in this paper are the following.

Theorem 3 (*Existence result*). *Suppose that f satisfis (Hf) and assume that one of the hypotheses (Hi) holds for $i = \overline{1,3}$, then, there exists a constant $\Lambda_* > 0$ such that the problem $(\mathscr{P}_\lambda)$ has a least two nontrivial solutions in $W^{1,2}_{\alpha,\mu}$ for any $\lambda \in]0, \Lambda_*[$.*

2.1. Proof of the main results

Existence of a First Solution

Step 1 : Existence of a Palais-Smale sequence

Lemma 4 *Let $f \in W^{*}\setminus\{0\}$ and suppose that one of the hypotheses (Hi) hold for $i=\overline{1,3}$. Then there exist positive numbers δ_1, ρ_1 and Λ_1 such that for all $\lambda \in]0,\ \Lambda_1[$ we have $I_\lambda(u)|_{\partial B_{\rho_1}(0)} \geq \delta_1 > 0$ and*

$$I_\lambda(u)|_{B_{\rho_1}(0)} \geq \begin{cases} -\frac{1}{2}\left(\left(\frac{b}{2}\right)^{\frac{-1}{2}}\lambda\|f\|_{W^*}\right)^2 & \text{if } (H1) \text{ is satisfied,} \\ -\frac{3}{4}\left(\left(\frac{a}{2}\right)^{\frac{-1}{4}}\lambda\|f\|_{W^*}\right)^{\frac{4}{3}} & \text{if one of } (Hi) \text{ is satisfied with } i=\overline{2,3}. \end{cases}$$

Step 2 : $u_n \to u$ in $W^{1,2}_{\alpha,\mu}$

Lemma 5 *Let $f \in W^{*}\setminus\{0\}$ and $(u_n) \subset W^{1,2}_{\alpha,\mu}$ be a Palais Smale sequence at level c_1 .for I_λ, then*

$$u_n \to u_1.$$

Existence of a Second Solution

Step 1 : The geometry conditions of the Mountain Pass theorem

Theorem 6 *For $\lambda \in (0,\lambda_1)$, we have*
(1) $I(0)=0$,
(2) there exist two numbers ρ and α such that $I(u) \geq \alpha$ for every $u \in \{u \in X : \|u\| = \rho\}$,
(3) there exist $v \in V$ such that $I(v) < \alpha$ and $\|v\| \geq \rho$.

Step 2 : The level of the energy functional

Lemma 7 *Suppose that f satisfies (Hf) and assume that one of the hypotheses (Hi) holds for $i=\overline{1,3}$. Then there exists $\Lambda_* > 0$ such that*

$$\sup_{t\geq 0} I_\lambda(tV_\varepsilon) < C_* + c_1, \ \text{ for all } \lambda \in]0,\Lambda_*[.$$

Step 3 : Palais Smale condition

Lemma 8 *Let $(u_n) \subset W^{1,2}_{\alpha,\mu}$ a Palais Smale sequence for I_λ for some $c \in \mathbb{R}^+$, then*

$$\text{either } u_n \to u_2 \ \text{ or } c \geq I_\lambda(u_2) + C_*.$$

3. CONCLUSION

In this paper we give a positive aftermath by proving the existence of of a weak solution to our problem. The key point is our technique is based on variational methods and concentration compactness principle.

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