MULTIPLE SOLUTIONS FOR NONHOMOGENEOUS NONLOCAL ELLIPTIC PROBLEMS WITH SINGULAR POTENTIALE

A. MATALLAH[1]*, S. BENMANSOUR*[2] *and A. RIMOUCHE*[3]

[(1,2)] Laboratoire d'analyse et controle des équations aux dérivées partielles.
Université Djilali Liabes. Sidi Bel Abès - Algeria
Ecole supérieure de management de Tlemcen, Algeria
[3] Laboratoire Systèmes Dynamiques et Applications.
Université de Tlemcen, Algeria

ABSTRACT

A class of nonlocal elliptic problems containing both singular term and nonhomogeneous nonlinearity is considered in a bounded domain in $\mathbb{R}^3$: the existence of two distinct solutions is obtained by the Ekeland Variational Principle and the Nehari decomposition.

1. INTRODUCTION

This paper deals with the existence and multiplicity of solutions for the following problem with Dirichlet boundary value :

$$(\mathscr{P}_\lambda)\begin{cases} -(b\int_\Omega \left(|\nabla u|^2 - \mu \dfrac{u}{|x|^2}\right) + a)(\Delta u - \mu \dfrac{u}{|x|^2}) = |u|^{p-2}u + \lambda f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $4 < p < 6$, a, b, λ, μ are positive constants with $\mu < \overline{\mu} = \frac{1}{4}$ and f belongs to H_μ^{-1} which is the dual of $H_\mu := H_\mu(\Omega)$ for $0 \leq \mu < \overline{\mu} = \frac{1}{4}$ equipped with the norm

$$\|u\|_\mu^2 := \int_\Omega \left(|\nabla u|^2 - \mu \frac{u^2}{|x|^2}\right) dx.$$

This problem is related to the following well known Hardy inequality [10] :

$$\left(\int_\Omega \frac{u^2}{|x|^2} dx\right)^{1/2} \leq 1/\overline{\mu} \left(\int_\Omega |\nabla u|^2 \, dx\right)^{1/2} \text{ for all } u \in C_0^\infty(\Omega),$$

The appearence of the integral $(b\int_\Omega |\nabla u|^2 dx + a)$,over the entire domain Ω implies that the equation in $(\mathscr{P}_\lambda)$ is no longer a pointwise identity then,the problem under consideration turns to be nonlocal.

It is well known that nonlocal problems serve to model several physical and biological phenomena for example when an elastic string with fixed ends is subjected to transverse vibrations; Kirchhoff in [12] was the first who proposed this study as an extension of the D'Alembert wave equation.

Without singularity, the problem $(\mathscr{P}_\lambda)$ reduces to the stationary elliptic version of Kirchhoff type problems ; this class attracted the interest of researchers and many existence results have been established see [1], [13], [3], [2] and the references therein. In the literature, there are many works where the autors obtained positive solutions for such problems by variational methods see [5] however in the situation where this approach doen't apply, othor methods as numerical ones can be sollicited ; interested readers can refer to [4], [8] and [15].

We begin by giving an overview about the previous research on the problem $(\mathscr{P}_\lambda)$.

For $b = \mu = 0$, $a = 1$, and $p = 2^* = \frac{2N}{N-2}$, $N \geq 3$,Tarantello [14] proved the existence of at least two solutions in a bounded domain of $\mathbb{R}^N$, under a suitable condition on f.

In the case $\mu = 0$ and $p = 2^* = \frac{2N}{N-2}$, Benmansour and Bouchekif [6] have shown under a sufficient condition on f, the existence of multiple solutions for $(\mathscr{P}_\lambda)$.

In the regular case $(\mu = 0)$, Benmansour and Matallah [7] obtained a multiplicity result to the problem $(\mathscr{P}_\lambda)$ via variational methods.

Before stating our main result, we give some notations that will be used in the remainder of this paper.

$\int u = \int_\Omega u dx$, $||u||_{H_\mu^-} = ||u||_-$, $|u|_p = (\int_\Omega |u|^p)^{1/p} dx$ are the norms in $H_\mu^{-1}$and L^p for $1 \leq p < \infty$ respectively, for $4 < p < 6$ we define the best constant S_p by

$$S_p := \inf_{H_\mu} \frac{\|u\|_\mu^2}{(\int_\Omega |u|^p dx)^{2/p}},$$

, B_c^r is the ball of center c and radius r, $o_n(1)$ denotes any quantity which tends to zero as n goes to infinity.

Before giving our main result, let us define

$$\begin{aligned} \lambda_1 &= \frac{\sqrt{ab(p-4)}}{\sqrt{2}(p-1)\|f\|_-}\left(\frac{2S_p^{p/2}\sqrt{3ab}}{(p-1)}\right)^{2/(p-2)}, \\ \lambda_2 &= \frac{a(p-2)}{2(p-1)\|f\|_-}\left(\frac{aS_p^{p/2}}{(p-1)}\right)^{(p-1)} \end{aligned}$$

and $\lambda_* = \max(\lambda_1, \lambda_2)$.

The main result can be described as follows.

Theorem 1 *For all $0 < \lambda < \lambda_*, \mu < \overline{\mu} = \frac{1}{4}$ the problem $(\mathscr{P}_\lambda)$ admits at least two positive solutions.*

This paper is organized as following : in the next section, we give the definition of Palais-Smale condition and some preliminaries which we will use later. We give the proof of our main result in Section 3.

2. SOME PRELIMINARY RESULTS

We define the energy functional associated to the problem $(\mathscr{P}_\lambda)$ as follows

$$I_\lambda(u) = \frac{b}{4}\|u\|_\mu^4 + \frac{a}{2}\|u\|_\mu^2 - \frac{1}{p}|u|_p^p - \lambda \int fu, \text{ for all } u \in H_\mu$$

where $u^+ = \max(0, u)$. In general a function $u \in H_\mu$ is said to be a weak solution of $(\mathscr{P}_\lambda)$ if it satisfies

$$\left(b||u||^2 + a\right) \int \left(\nabla u \nabla v - \mu \frac{uv}{|x|^2}\right) - \int |u|^{p-2} uv - \lambda \int fv = 0, \text{for all } v \in H_\mu.$$

To overcome the fact that the energy functional I_λ is not bounded from below in H_μ and in order to obtain multiplicity result, we introduce the Nehari manifold defined by

$$\mathcal{N}_\lambda = \left\{u \in H_\mu \backslash \{0\} : \left\langle I'_\lambda(u),\, u\right\rangle = 0\right\}$$

which we split into three subsets :.

$$\mathcal{N}_\lambda^+ := \left\{u \in \mathcal{N}_\lambda : h''_u(1) > 0\right\},$$
$$\mathcal{N}_\lambda^0 := \left\{u \in \mathcal{N}_\lambda : h''_u(1) = 0\right\}$$

and

$$\mathcal{N}_\lambda^- := \left\{u \in \mathcal{N}_\lambda : h''_u(1) < 0\right\},$$

where $h_u(t) = I_\lambda(tu)$, $t \geq 0$, for more details about these maps, one can see [11].

We give the following result whose proof is similar than the one given in [9].

Lemma 2 *Suppose that u_0 is a local minimiser of I_λ in $\mathcal{N}_\lambda$ and $u_0 \notin \mathcal{N}_\lambda^0$, then $I'_\lambda(u_0) = 0$ in H_μ^{-1}.*

In preparation for the proof of the main result, we need the following lemmas.

Lemma 3 *For all $0 < \lambda < \lambda_*, \mu < \frac{1}{4}$ and each $u \in H_\mu \diagdown \{0\}$, there exists unique $t^u_{\max} > 0$ such that*

i)If $\int f|u| \leq 0$, then there exists $t^+ = t^+(u) \neq 0$ such that $t^+ u \in \mathcal{N}_\lambda^-$ and $I_\lambda(t^+u) = \max_{t\geq 0} I_\lambda(tu)$.

ii)If $\int f|u| > 0$, then there exists $t^+ = t^+(u)$ and $t^- = t^-(u)$ such that $t^+u \in \mathcal{N}_\lambda^-$, $t^-u \in \mathcal{N}_\lambda^+$ and $I_\lambda(t^+u) = \max_{t\geq t^u_{\max}} I_\lambda(tu)$, $I_\lambda(t_1^- u) = \min_{0\leq tt \leq t^u_{\max}} I_\lambda(tu)$.

Lemma 4 *For all $0 < \lambda < \lambda_*, \mu < \frac{1}{4}$, we have $\mathcal{N}_\lambda^0 = \varnothing$.*

Lemma 5 *For all $0 < \lambda < \lambda_*, \mu < \frac{1}{4}$ and each $u \in \mathcal{N}_\lambda$, there exist $\varepsilon > 0$ and a differentiable function $t : B(0,\varepsilon) \subset H \longrightarrow \mathbb{R}^+$ such that $t(0) = 1$, $t(v)(u - v) \in \mathcal{N}_\lambda$ for $\|v\| < \varepsilon$.*

Lemma 6 *The functional I_λ is coercive and bounded from below on $\mathcal{N}_\lambda$.*

It is known that for all $u \in \mathcal{N}_\lambda$, we have

Lemma 7 *For all $0 < \lambda < \lambda_*$, then there exist two minimising sequences $(u_n) \subset \mathcal{N}_\lambda^+$ and $(v_n) \subset \mathcal{N}_\lambda^-$ such that*

i) $I(u_n) < c_0 + \frac{1}{n}$ and $I(w_1) \geq I(u_n) - \frac{1}{n}\|w_1 - u_n\|_\mu$ for all $w_1 \in \mathcal{N}_\lambda^+$.

ii) $I(v_n) < c_1 + \frac{1}{n}$ and $I(w_2) \geq I(v_n) - \frac{1}{n}\|w_2 - v_n\|_\mu$ for all $w_2 \in \mathcal{N}_\lambda^-$.

Proof. It is clear that I_λ is bounded on $\mathcal{N}_\lambda$, then applying the Ekeland varitional principle, we obtain two minimising sequences $(u_n) \subset \mathcal{N}_\lambda^+$ and $(v_n) \subset \mathcal{N}_\lambda^-$ which verify (i) and (ii) respectively. ■

Before giving the proof of our theorem, we show a compactness result related to the Palais-Smale condition. For this let us recall that (u_n) is said to be a Palais-Smale sequence for I_λ if

$I_\lambda(u_n)$ is bounded on H and $I'_\lambda(u_n) \to 0$ in H_μ^-.

Lemma 8 *If (w_n) is a Palais-Smale sequence of I_λ bounded in H_μ, then (w_n) has a subsequence in H_μ which is strongly convergente.*

3. PROOF OF THEOREM 1

Define $m_{\lambda}^{+}=\inf_{u\in\mathcal{N}_{\lambda}^{+}} I_{\lambda}(u)$ and $m_{\lambda}^{-}=\inf_{v\in\mathcal{N}_{\lambda}^{-}} I_{\lambda}(v)$ and we claim that : $m_{\lambda}^{+}<0$ and in particular $m_{\lambda}^{+}=\inf_{v\in\mathcal{N}} I_{\lambda}(v)$.

Indeed, for $u\in\mathcal{N}_{\lambda}^{+}$ we have

$$\begin{aligned}\lambda(p-1)\int fu &> b(p-4)\|u\|_{\mu}^{4}+a(p-2)\|u\|\mu^{2}\\ &\geq a(p-2)\|u\|_{\mu}^{2},\end{aligned}$$

then,

$$\begin{aligned}I_{\lambda}(u) &= \frac{b}{4}\|u\|_{\mu}^{4}+\frac{a}{2}\|u\|_{\mu}^{2}-\frac{1}{p}|u|_{p}^{p}-\lambda\int fu\\ &= \frac{b}{4}\|u\|_{\mu}^{4}+\frac{a}{2}\|u\|_{\mu}^{2}-\frac{1}{p}(b\|u\|_{\vartheta}^{4}+a\|u\|_{\mu}^{2})-\lambda\frac{(p-1)}{p}\int fu\\ &\leq \frac{a(p-2)}{2p}\|u\|_{\mu}^{2}-\lambda\frac{(p-1)}{p}\int fu\\ &\leq -\frac{a(p-2)}{2p}\|u\|_{\mu}^{2}<0\end{aligned}$$

then, we can conclude that $m_{\lambda}^{+}<0$.

From Lemma 5, we obtain the existence of two minimising sequences $(u_n)\subset\mathcal{N}_{\lambda}^{+}$ and $(v_n)\subset\mathcal{N}_{\lambda}^{-}$ which from Lemma 6 converge strongly to u_{λ}^{+} and u_{λ}^{-} respectively, then $u_{\lambda}^{+}\in\mathcal{N}_{\lambda}^{+}$ and $u_{\lambda}^{-}\in\mathcal{N}_{\lambda}^{-}$ with $I_{\lambda}\left(u_{\lambda}^{+}\right)=m_{\lambda}^{+}$ and $I_{\lambda}\left(u_{\lambda}^{-}\right)=m_{\lambda}^{-}$.From the fact that $I_{\lambda}(|u|)=I_{\lambda}(u)$ for all $u\in H_{\mu}$, we get two positive distinct solutions to the problem $(\mathscr{P}_{\lambda})$.

4. CONCLUSION

In this work, we prove a multiplicity result for a nonlocal elliptic problems with singular term and nonhomogeneous nonlinearity is considered in a bounded domain in $\mathbb{R}^3$ by the Ekeland Variational Principle and the Nehari decomposition.

5. REFERENCES

[1] C.O. Alves, F J.S.A. Correa, T.F. Ma, Positive solutions for a quasilinear elliptic equation of Kirchhoff type, Comput. Math. Appl. 49 (2005) 85-93.

[2] P. D'Ancona,Y. Shibata, On global solvability of nonlinear viscoelastic equations in the analytic category, Math. Methods Appl. Sci. 17 (1994) 477-489.

[3] P. D'Ancona, S. Spagnolo, Global solvability for the degenerate Kirchhoff equation with real analytic data, Invent. Math. 108 (1992) 247-262.

[4] R. Abazari, K. Yildirim ; Numerical study of Sivashinsky equation using a splitting scheme based on Crank-Nicolson method, Mathematical Methods in the Applied Sciences 42 (16), 5509-5521.

[5] H. Benchira, S. Guendouz, A. Matallah, Solutions for singular Kirchhoff problem involving critical nonlinearity, 7(1) (2019), 74-81.

[6] S. Benmansour, M. Bouchekif ; Nonhomogeneous elliptic Kirchhoff type problems involving critical Sobolev exponent, Electon. J. Diff. Equ. vol. 2015 (2015), No. 69, pp. 1-11.

[7] S. Benmansour, A. Matallah, Multiple solutions for non homogeneous elliptic problems of Kirchhoff type. IEEE (ICEMIS), pp(1-4). DOI : 10. 1109, 7745374 (2016), pp. 1-4.

[8] M. Bayram, K. Yıldırım ; Approximates Method for Solving an Elasticity Problem of Settled of the Elastic Ground with Variable Coefficients, Natural Sciences Publishing (NSP).

[9] K.J. Brown, Y. Zhang, The Nehari manifold for a semilinear elliptic equation with a sign-changing weight functions, J. Differential Equations. 193 (2003) 481-499.

[10] K.S. Chou, C.W. Chu, On the best constant for a weighted Sobolev-Hardy Inequality, J. London Math. Soc. 2, 137-151 (1993).

[11] Y. Chen, Y.C. Kuo, T.F. Wu, The Nehari manifold for a Kirchhoff type problems with critical exponent functions, J. Differential Equations. 250 (2011) 1876-1908.

[12] G. Kirchhoff, Mechanik. Teubner. Leipzig (1883).

[13] T.F. Ma, J.E. Munoz Rivera, Positive solutions for a nonlinear elliptic transmission problem, Appl. Math. Lett. 16(2) (2003) 243-248.

[14] G. Tarantello ; On nonhomogeneous elliptic equations involving critical Sobolev exponent, Ann. Inst. Henri Poincaré Anal. Nonlinéaire 9 (1992) 281-304.

[15] K. Yıldırım, B. Ibis, M. Bayram ; New solutions of the nonlinear Fisher type equations by the reduced differential transform, Nonlinear. Sci. Lett. A 3 (1) 29-36.