A MODIFIED WRIGHT FUNCTION FOR CERTAIN GENERALIZED FRACTIONAL OPERATORS

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ABSTRACT

The paper is devated to the study of a modified Wright function for certain generalized fractional operators,nowadays known as $B_{\nu,\rho}(z)$. Conditions for the existence of $B_{\nu,\rho}(z)$ are proved. The representation of the function in terms of the general Wright function denoted by $W_{\lambda,\mu}(z)$ and of the two related auxiliary function $F_{\nu}(z)$, $M_{\nu}(z)$ which depend on asingle parameter. Special cases, involving the Mittag−Leffler function and its generalizations, are considered. The obtained results imply more precisely the known results.

Keywords : modied Wright function, generalized fractional, $\rho-$Laplace transform, Mittag−Leffler types function.

1. INTRODUCTION

Fractional calculus is a natural extension of ordinary calculus, where integrals and derivatives are defined for arbitrary real orders. Since $17th$ century, when fractional calculus was born, several different derivatives have been introduced : Riemann−Liouville, Hadamard, Grunwald−Letnikov, Caputo, just to mention a few [24, 26, 31], each of them with its own advantages and disadvantages. Recently, U. Katugampola presented new types of fractional operators, which generalize both the Riemann−Liouville and Hadamard fractional operators [28, 29, 30, 25]. The authors in [27] did define the Caputo version of the generalized fractional derivatives, which generalizes the concept of Caputo and Caputo−Hadamard fractional derivatives. In [25], the authors giveen relation between generalized fractional derivatives and Caputo version of the generalized fractional derivatives.

The special functions of mathematical physics are found to be very useful for finding solutions of initial and/or boundary value problems governed by partial differential equations and fractional differential equations. Special functions have widespread applications in other areas of mathematics and often new perspectives in special functions are motivated by such connections. Several special functions, called recently special functions of fractional calculus, play a very important and interesting role as solutions of fractional order differential equations, suchas the Mittag−Leffler function, Wright function with its auxiliary functions, and Fox's −function. The Wright function is one of the special functions which plays an important role in the solution of linear partial fractional differential equations. It was introduced for the first time in [3, 4] in connection with a problem in the number theory regarding the asymptotic of the number of some special partitions of the natural numbers. Recently this function has appeared in papers related to partial differential equations of fractional order. Considering the boundary value problems for the fractional diffusion−wave equation, that is, the linear partial integrodifferential equation obtained from the classical diffusion or wave equation by replacing the first. Furthermore,

extending the methods of Lie groups in partial differential equations to the partial differential equations of fractional order, it was shown that some of the group−invariant solutions of these equations can be given in terms of the Wright and the generalized Wright functions [11].A list of formulas concerning this function can be found in the hand book of Bateman Project,Erdélyi et al. 1953[12].
The Wright function, that we denote by $W_{\nu,\mu}(z)$, is so named in honour of E. Maitland Wright, the eminent British mathematician, who introduced and investigated this function in a series of notes starting from 1933 in the framework of the asymptotic theory of partitions, see [4, 5, 7]. The function is dened by the series representation, convergent in the whole $z-$complex plane,

$$W_{\nu,\mu}(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!\Gamma(\nu n+\mu)};\ \nu > -1;\ \mu \in \mathbb{C}. \tag{1}$$

Originally, Wright assumed $\lambda \geqslant 0$, and, only in 1940[3], he considered $1 < \nu < 0$.
Mainardi, in his rst analysis of the time−fractional diusion equation [13, 15], aware of the Bateman handbook [16], but not yet of the 1940 paper by Wright [3], introduced the two (Wright−type) entire auxiliary functions,

$$F_\nu(z) := W_{-\nu,0}(-z);\ 0 < \nu < 1, \tag{2}$$

and

$$M_\nu(z) := W_{-\nu,1-\nu}(-z);\ 0 < \nu < 1, \tag{3}$$

inter−related through

$$F_\nu(z) = \nu z M_\nu(z). \tag{4}$$

As a matter of fact, functions $F(z)$ and $M(z)$ are particular cases of the Wright function of the second kind $W_{\lambda,\mu}(z)$ by setting $\lambda =$ and $\mu = 0$ or $\mu = 1$, respectively.
In a continuation of this study, we investigate the generalized Wright function $B_{\nu,\rho}(z)$ which is defined for $0 < \nu < 1$, $\rho > 0$ and $z \in \mathbb{C}$, as :

$$B_{\nu,\rho}(z) = \sum_{n=0}^{\infty} \frac{\left(-\frac{z^\rho}{\rho}\right)^n}{n!\Gamma(1-\nu-\nu n)}; \tag{5}$$

where $\Gamma(.)$ is a gamma function.

2. PRELIMINARIES

Some basic definitions, theorems, lemmas and assumptions which which will be use later.

The generalized fractional integrals are defined by, $n-1 < \alpha \leq n$; see [25, 28]

$$\left(\mathscr{I}_{a^+}^{\alpha,\rho}\right) g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\alpha-1} g(\tau) \frac{\mathrm{d}\tau}{\tau^{1-\rho}},$$

where $\Gamma(.)$ is the Euler gamma function.

The left generalized fractional derivatives of order $\alpha > 0$ are defined by

$$\left(\mathscr{D}_{a^+}^{\alpha,\rho}\right) g(t) = \frac{\gamma^n}{\Gamma(n-\alpha)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{n-\alpha-1} g(\tau) \frac{\mathrm{d}\tau}{\tau^{1-\rho}},\ t \in [a,b]$$

The left-sided generalized Caputo−type fractional derivative of g of order α is defined

$$\left({}^c\mathscr{D}_{a^+}^{\alpha,\rho}\right) g(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{n-\alpha-1} \frac{(\gamma^n g)(\tau)\mathrm{d}\tau}{\tau^{1-\rho}}.$$

where $\gamma = t^{1-\rho}\frac{\mathrm{d}}{\mathrm{d}t}$.

For $\alpha > 0;\ \beta > 0;\ 1 \leqslant p < \infty;\ a \in (0,\infty);\ \rho,\ c \in \mathbb{R};\ \rho \geqslant c$.

$$\mathscr{I}_{a^+}^{\alpha,\rho}\, \mathscr{I}_{a^+}^{\beta,\rho} g = \mathscr{I}_{a^+}^{\alpha+\beta,\rho} g;\ g \in \mathbb{X}_c^p(a,b).$$

The Mittag−Leffler function is an important function that finds widespread use in the world of fractional calculus. Just as the exponential naturally arises out of the solution to integer order differential equations, the Mittag−Leffler function plays ananalogous role in the solution of non−integerorder differential equations. Infact, the exponential function itself is a very special form, one of an infinite set of the se seemingly ubiquitous functions. The standard definition of Mittag−Leffler [1, 2, 8, 10, 19, 21, 18, 9, 14] is given as follows :

$$E_v(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(vk+1)};\ v \in \mathbb{C},\ \Re e(\alpha) > 0,\ z \in \mathbb{C} \tag{6}$$

A two parameter function of the Mittag−Leffler type is dened by the series expansion

$$E_{v,w}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(vk+w)};\ v,w \in \mathbb{C},\ \Re e(\alpha) > 0,\ \Re e(w) > 0,\ z \in \mathbb{C} \tag{7}$$

Also the function $E_{\alpha,\beta}(z)$ has the integral representation

$$E_{\alpha,\beta}(z) = \frac{1}{2\pi i} \int_{\mathscr{C}} \frac{t^{\alpha-\beta} e^t}{t^\beta - z} \mathrm{d}t,$$

where the path of integration $\mathscr{C}$ is a loop which starts and ends at $-\infty$, and encircles the circles disc $|t| \leq |z|^{1/\beta}$ in the positive sense : $|\arg(t)| \leq \pi$ on $\mathscr{C}$, we shortly write $E_{\alpha,1}(z) = E_\alpha(z)$.

In our analysis we will make extensive use of integral transforms of $\rho-$Laplace. We do not point out the conditions of validity and the main rules, since they are given in any textbook on advanced mathematics. see [20].

Let $g : [0,\infty) \longrightarrow \mathbb{R}^+$ be a real valued function. The ρ-Laplace transform of g is defined by

$$\mathscr{L}_\rho \left\{g(t)\right\}(\tau) = \int_0^{+\infty} \exp\left(-\tau \frac{t^\rho}{\rho}\right) g(t) \frac{\mathrm{d}t}{t^{1-\rho}},$$

for all values of τ.

Let $\alpha > 0$ and g be a piecewise continuous function on each interval $[0,t]$ and of ρ exponential order $e^{c\frac{t^\rho}{\rho}}$. Then the Laplace transform formula for the Caputo type generalized fractional integral is defined by

$$\mathscr{L}_\rho \left\{ \mathscr{I}_{a^+}^{\alpha,\rho} g(t) \right\}(\tau) = \tau^{-\alpha} \mathscr{L}_\rho \left\{g(t)\right\},\ \tau > c.$$

3. MAIN RESULTS

The integral representations for the $B_{\alpha,\rho}-$Wright function read

$$B_{v,\rho}(z) = \frac{1}{2\pi i} \int_{H_a} e^{\sigma - \frac{z^\rho}{\rho}\sigma^\alpha} \frac{d\sigma}{\sigma^{1-v}};\ 0 < v < 1;\ \mu \in \mathbb{C} \tag{8}$$

where H_a denotes the Hankel path. We remind that the Hankel path is a loop that starts from $-\infty$ along the lower side of the negative real axis, encircles the circular area around the origin with radius $\varepsilon \longrightarrow 0$ in the positive sense, and ends at $-\infty$ along the upper side of the negative real axis. The equivalence of the series and integral representations is easily proved using Hankel formula for the Gamma function

$$\frac{1}{\Gamma(\xi)} = \frac{1}{2\pi i} \int_{H_a} e^u u^{-\xi} du;\ \xi \in \mathbb{C}. \tag{9}$$

and performing a term-by-term integration. In fact,

$$\begin{aligned} B_{v,\rho}(z) &= \frac{1}{2\pi i}\int_{H_a} e^{\sigma-\frac{z^\rho}{\rho}\sigma^\alpha}\frac{d\sigma}{\sigma^{1-v}} \\ &= \frac{1}{2\pi i}\int_{H_a} e^{\sigma}\sum_{n=0}^{\infty}\frac{\left(-\frac{z^\rho}{\rho}\right)^n\sigma^{vn}}{n!}\frac{d\sigma}{\sigma^{1-v}} \\ &= \sum_{n=0}^{\infty}\frac{\left(-\frac{z^\rho}{\rho}\right)^n}{n!}\frac{1}{2\pi i}\int_{H_a}\frac{e^\sigma}{\sigma^{1-v-vn}}d\sigma \\ &= \sum_{n=0}^{\infty}\frac{\left(-\frac{z^\rho}{\rho}\right)^n}{n!}\frac{1}{\Gamma(1-v-vn)} \end{aligned}$$

The series representations for the $B_{v,\rho}-$Wright functions read

$$B_{v,\rho}(z) := \sum_{n=0}^{\infty}\frac{\left(-\frac{z^\rho}{\rho}\right)^n}{n!\Gamma(1-v-vn)} = \frac{1}{\pi}\sum_{n=1}^{\infty}\left(-\frac{z^\rho}{\rho}\right)^n\frac{1}{(n-1)!}\Gamma(vn)\Gamma(nv\pi).$$

by using the reection formula for the Gamma function $\Gamma(\zeta)\Gamma(1-\zeta) = \pi/\sin\pi\zeta$.
Furthermore, we have $B_{v,\rho}(0) = 1/\Gamma(1-v)$.
Explicit expressions of $B_{v,\rho}$ in terms of known functions are expected for some particular values of . . In the particular cases $v = 1/2$ and $v = 1/3$ we find

$$B_{1/2,\rho}(z) = \frac{1}{\sqrt{pi}}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{2}\right)_n\frac{\left(\frac{z^\rho}{\rho}\right)^{2n}}{(2n)!} = \frac{1}{\sqrt{pi}}exp\left(-\left(\frac{z^\rho}{\rho}\right)^2/4\right)$$

and

$$\begin{aligned} B_{1/3,\rho}(z) &= \frac{1}{\Gamma(2/3)}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)_n\frac{\left(\frac{z^\rho}{\rho}\right)^{3n}}{(3n)!} - \frac{1}{\Gamma(1/3)}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)_n\frac{\left(\frac{z^\rho}{\rho}\right)^{3n+1}}{(3n+1)!} \\ &= 3^{2/3}Ai\left(\left(\frac{z^\rho}{\rho}\right)/3^{1/3}\right) \end{aligned}$$

where Ai denotes the Airy function.
Moreover, the analysis of the limiting cases $\rho = 1$, $v = 0$ and $v = 1$ requires special attention.

$$B_{v,1}(z) = M_v(z)$$

and

$$B_{0,\rho}(z) = e^{-\left(\frac{z^\rho}{\rho}\right)}$$

The limiting case $v = 1$ is singular for both the auxiliary functions as expected from the definition of the general Wright function when $\lambda = -v = -1$. Later we will deal with this singular case for the $B_{v,\rho}$Wright function when the variable is real and positive.
Let us state some relevant properties of the $B_{v,\rho}$ function in view of its role in time$-$fractional diusion processes.

Lemma 1 *We start with the $\rho-$Laplace transform pairs involving exponentials in the $\rho-$Laplace domain.*

$$e^{-\lambda^v}\lambda^{v-1} = \int_0^\infty\left(\frac{r^\rho}{\rho}\right)^{-v}B_{v,\rho}\left(\frac{r^{-v}}{\rho^{\frac{-v-1}{\rho}}}\right)e^{-\lambda\frac{r^\rho}{\rho}}\frac{dr}{r^{1-\rho}} \tag{10}$$

Proof.

$$
\begin{aligned}
& \int_0^\infty \lambda^{1-\nu}\left(\frac{r^\rho}{\rho}\right)^{-\nu} B_{\nu,\rho}\left(\frac{r^{-\nu}}{\rho^{\frac{-\nu-1}{\rho}}}\right) e^{-\lambda\frac{r^\rho}{\rho}}\frac{dr}{r^{1-\rho}}\\
= & \int_0^\infty \lambda^{1-\nu}\left(\frac{r^\rho}{\rho}\right)^{-\nu}\left[\sum_{n=0}^\infty \frac{(-1)^n\left(\frac{r^\rho}{\rho}\right)^{-\nu n}}{n!\Gamma(1-\nu-\nu n)}\right] e^{-\lambda\frac{r^\rho}{\rho}}\frac{dr}{r^{1-\rho}}\\
= & \sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(1-\nu-\nu n)}\int_0^\infty \lambda^{1-\nu}\left(\frac{r^\rho}{\rho}\right)^{-\nu-\nu n} e^{-\lambda\frac{r^\rho}{\rho}}\frac{dr}{r^{1-\rho}}\\
= & \sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(1-\nu-\nu n)}\int_0^\infty \lambda^{1-\nu}\left(\frac{x}{\lambda}\right)^{-\nu-\nu n} e^{-x}\frac{dx}{\lambda}\\
= & \sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(1-\nu-\nu n)}\lambda^{\nu n}\int_0^\infty e^{-x}x^{-\nu-\nu n}dx\\
= & \sum_{n=0}^\infty \frac{(-1)^n}{n!\Gamma(1-\nu-\nu n)}\lambda^{\nu n}\Gamma(1-\nu-\nu n)\\
= & \sum_{n=0}^\infty \frac{(-\lambda^\nu)^n}{n!} = e^{-\lambda^\nu}
\end{aligned}
$$

and on a

$$
e^{-\lambda^\nu} = \int_0^\infty \nu\left(\frac{r^\rho}{\rho}\right)^{-\nu-1} B_{\nu,\rho}\left(\frac{r^{-\nu}}{\rho^{\frac{-\nu-1}{\rho}}}\right) e^{-\lambda\frac{r^\rho}{\rho}}\frac{dr}{r^{1-\rho}} \tag{11}
$$

in article [22], on

$$
\int_0^\infty \nu t^{-1-\nu} M_\nu(t^{-\nu}) e^{-\lambda t} dt = e^{-\lambda^\alpha}
$$

on faire le changement de variable $t = \frac{r^\rho}{\rho}$

$$
\begin{aligned}
e^{-\lambda^\alpha} &= \int_0^\infty \nu t^{-1-\nu} M_\nu(t^{-\nu}) e^{-\lambda t} dt\\
&= \int_0^\infty \nu\left(\frac{r^\rho}{\rho}\right)^{-1-\nu} M_\nu\left(\left(\frac{r^\rho}{\rho}\right)^{-\nu}\right) e^{-\lambda\frac{r^\rho}{\rho}}\frac{dr}{r^{1-\rho}}\\
&= \int_0^\infty \nu\left(\frac{r^\rho}{\rho}\right)^{-\nu-1} B_{\nu,\rho}\left(\frac{r^{-\nu}}{\rho^{\frac{-\nu-1}{\rho}}}\right) e^{-\lambda\frac{r^\rho}{\rho}}\frac{dr}{r^{1-\rho}}
\end{aligned}
$$

■ Let us point out the asymptotic behaviour of the function $B_{\alpha,\rho}(r)$ when $r \longrightarrow 1$. Choosing as a variable r/ν rather than r, the computation of the desired asymptotic representation by the saddle−point approximation is straightforward. Mainardi and Tomirotti [17] have obtained

Lemma 2 *From the above considerations we recognize that, for the $B_{\nu,\rho}$−Wright functions, the following rule for absolute moments in $\mathbb{R}^+$ holds*

$$
\int_0^\infty \left(\frac{r^\rho}{\rho}\right)^\delta B_{\nu,\rho}(r)\frac{dr}{r^{1-\rho}} = \frac{\Gamma(\delta+1)}{\Gamma(\nu\delta+1)};\ \delta > -1;\ 0 \le \alpha < 1. \tag{12}
$$

Proof. In order to derive this fundamental result, we proceed as follows on the basis of the integral representation (8) :

$$\begin{aligned}\int_0^{\infty}\left(\frac{r^{\rho}}{\rho}\right)^{\delta}B_{v,\rho}(r)\frac{dr}{r^{1-\rho}} &= \int_0^{\infty}\left(\frac{r^{\rho}}{\rho}\right)^{\delta}\left[\frac{1}{2\pi i}\int_{H_a}e^{\sigma-\frac{r^{\rho}}{\rho}\sigma^{v}}\frac{d\sigma}{\sigma^{1-v}}\right]\frac{dr}{r^{1-\rho}}\\ &= \frac{1}{2\pi i}\int_{H_a}e^{\sigma}\left[\int_0^{\infty}e^{-\frac{r^{\rho}}{\rho}\sigma^{v}}\left(\frac{r^{\rho}}{\rho}\right)^{\delta}\frac{dr}{r^{1-\rho}}\right]\frac{d\sigma}{\sigma^{1-v}}\\ &= \frac{\Gamma(\delta+1)}{2\pi i}\int_{H_a}e^{\sigma}\sigma^{-(v\delta+1)}d\sigma=\frac{\Gamma(\delta+1)}{\Gamma(v\delta+1)}.\end{aligned}$$

Above we have legitimized the exchange between integrals and used the identity

$$\int_0^{\infty}e^{-\frac{r^{\rho}}{\rho}\sigma^{v}}\left(\frac{r^{\rho}}{\rho}\right)^{\delta}\frac{dr}{r^{1-\rho}}=\frac{\Gamma(\delta+1)}{(\sigma^{v})^{v\delta+1}},$$

along with the Hankel formula of the Gamma function. ■

We now point out that the $B_{v,\rho}-$Wright function is related to the Mittag$-$Leffler function through the following $\rho-$Laplace transform,

$$B_{v,\rho}(r)\ \underset{\longrightarrow}{\mathscr{L}_{\rho}}\ E_{v}(-s);\ 0<v<1. \tag{13}$$

For the reader's convenience we provide a simple proof of (13) by using two dierent approaches. We assume that the exchanges between integrals and series are legitimate in view of the analyticity properties of the involved functions. In the first approach we use the integral representations of the two functions obtaining

$$\begin{aligned}\int_0^{\infty}e^{-s\frac{r^{\rho}}{\rho}}B_{v,\rho}(r)\frac{dr}{r^{1-\rho}} &= \int_0^{\infty}e^{-s\frac{r^{\rho}}{\rho}}\left[\frac{1}{2\pi i}\int_{H_a}e^{\sigma-\frac{r^{\rho}}{\rho}\sigma^{v}}\frac{d\sigma}{\sigma^{1-v}}\right]\frac{dr}{r^{1-\rho}}\\ &= \frac{1}{2\pi i}\int_{H_a}e^{\sigma}\left[\int_0^{\infty}e^{-\frac{r^{\rho}}{\rho}(s+\sigma^{v})}\frac{dr}{r^{1-\rho}}\right]\frac{d\sigma}{\sigma^{1-v}}\\ &= \frac{1}{2\pi i}\int_{H_a}\frac{e^{\sigma}\sigma^{v-1}}{\sigma^{v}+s}d\sigma=E_{v}(-s).\end{aligned}$$

In the second approach we develop in series the exponential kernel of the $\rho-$Laplace transform and we use the expression (12) for the absolute moments of the $B_{v,\rho}-$Wright function arriving to the following series representation of the Mittag$-$Leffler function,

$$\begin{aligned}\int_0^{\infty}e^{-s\frac{r^{\rho}}{\rho}}B_{v,\rho}(r)\frac{dr}{r^{1-\rho}} &= \int_0^{\infty}\sum_{n=0}^{\infty}\frac{\left(-s\frac{r^{\rho}}{\rho}\right)^{n}}{n!}B_{v,\rho}(r)\frac{dr}{r^{1-\rho}}\\ &= \sum_{n=0}^{\infty}\frac{(-s)^{n}}{n!}\int_0^{\infty}\left(\frac{r^{\rho}}{\rho}\right)^{n}B_{v,\rho}(r)\frac{dr}{r^{1-\rho}}\\ &= \sum_{n=0}^{\infty}\frac{(-s)^{n}}{n!}\frac{\Gamma(1+n)}{\Gamma(1+vn)}\\ &= \sum_{n=0}^{\infty}\frac{(-s)^{n}}{\Gamma(1+vn)}=E_{v}(-s).\end{aligned}$$

We note that the transformation term by term of the series expansion of the $B_{v,\rho}-$Wright function is not legitimate because the function is not of exponential order, see [6] . However, this procedure yields the formal asymptotic expansion of the Mittag$-$Leffler function $E_{v}(-s)$ as $s\longrightarrow\infty$

in a sector around the positive real axis, see e.g. [1, 2], that is

$$\begin{aligned}\int_0^\infty e^{-s\frac{r^\rho}{\rho}} B_{\nu,\rho}(r)\frac{dr}{r^{1-\rho}} &= \int_0^\infty e^{-s\frac{r^\rho}{\rho}} \sum_{n=0}^{\infty} \frac{\left(-\frac{r^\rho}{\rho}\right)^n}{n!\Gamma(-\nu n+(1-\nu))}\frac{dr}{r^{1-\rho}} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{\Gamma(-\nu n+(1-\nu))}\frac{1}{s^{n+1}} \\ &= \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{\Gamma(-\nu m+1)}\frac{1}{s^m}\sim E_\nu(-s).\end{aligned}$$

finlly, we see relation between a two parameter function of the Mittag−Leffler and $\rho-$Laplace transforms :

$$\begin{aligned}\int_0^\infty \nu\left(\frac{r^\rho}{\rho}\right) B_{\nu,\rho}(r)e^{-s\frac{r^\rho}{\rho}}\frac{dr}{r^{1-\rho}} &= \int_0^\infty \sum_{n=0}^{\infty}\frac{\nu\left(-z\frac{r^\rho}{\rho}\right)^n}{\Gamma(1+n)}\left(\frac{r^\rho}{\rho}\right) B_{\nu,\rho}(r)\frac{dr}{r^{1-\rho}} \\ &= \sum_{n=0}^{\infty}\frac{(-z)^n}{\Gamma(1+n)}\int_0^\infty \nu\left(\frac{r^\rho}{\rho}\right)^{1+n} B_{\nu,\rho}(r)\frac{dr}{r^{1-\rho}} \\ &= \sum_{n=0}^{\infty}\frac{(-z)^n}{\Gamma(1+n)}\nu\frac{\Gamma(1+n+1)}{\Gamma(1+(1+n)\nu)} \\ &= \sum_{n=0}^{\infty}\frac{(-z)^n}{\Gamma(\nu+\nu n)} = E_{\nu,\nu}(-z)\end{aligned}$$

4. CONCLUSIONS

The fundamental issue of the fractional operators and their generalized versions is to define them correctly of functions. In this paper, we defined the generalized $B_{\nu,\rho}-$Wright functions by which the fundamental solutions of these equations are expressed. We noticed that case as $\rho = 1$ will result in $M-$Wright function.

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