LOWER AND UPPER SOLUTIONS FOR CONFORMABLE FRACTIONAL DIFFERENTIAL EQUATIONS

Bouharket Bendouma

Faculty of Mathematics and Informatics, Ibn Khaldoun, Tiaret University, PoBox 78, 14000 zaâroura, Tiaret, Algeria

ABSTRACT

In this article, we study the existence of solutions to conformable fractional differential equations with periodic boundary condition. Existence results are obtained by the method of upper and lower solutions and Schauder's fixed point theorem.

1. INTRODUCTION

Fractional differential equations plays an important role in describing many phenomena and processes in various fields of science such as physics, chemistry, control systems, population dynamics, etc., see [3, 5, 6]. Recently, a new fractional derivative called the conformable fractional derivative was introduced by Khalil et al. [2].

In this paper, we are concerned with the existence of a solutions for the following conformable fractional differential equations with periodic boundary condition :

$$\begin{cases} x^{(\alpha)}(t) = f(t,x(t)), \qquad for\, t \in J = [a,b],\ a > 0, \\ x(a) = x(b). \end{cases} \tag{1}$$

Where $0 < \alpha \leq 1$, $f : J \times \mathbb{R} \to \mathbb{R}$ is a continuous function, $x^{(\alpha)}(t)$ denotes the conformable fractional derivative of x at t of order α.

The paper is organized as follows. In Section 2, we present the main concepts of conformable fractional derivative calculus and we give some useful preliminary results. In Section 3, we prove existence of solutions to problem (1) by using the method of upper and lower solutions and Schauder's fixed point theorem.

2. PRELIMINARIES

In this section, we introduce the definition of conformable fractional calculus and their important properties.

Definition 1 *[2] Given a function $f : [0,\infty) \to \mathbb{R}$ and a real constant $\alpha \in (0,1]$. The conformable fractional derivative of f of order α is defined by,*

$$f^{(\alpha)}(t) := \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon} \tag{2}$$

for all $t > 0$. If $f^{(\alpha)}(t)$ exists and is finite, we say that f is α-differentiable at t.

If f is α-differentiable in some interval $(0,a)$, $a > 0$, and $\lim_{t\to 0^+} f^{(\alpha)}(t)$ exists, then the conformable fractional derivative of f of order α at $t = 0$ is defined as

$$f^{(\alpha)}(0) = \lim_{t\to 0^+} f^{(\alpha)}(t).$$

Theorem 1 *[2] Let $\alpha \in (0,1]$ and $f : [0,\infty) \to \mathbb{R}$ a α-differentiable function at $t_0 > 0$, then f is continuous at t_0.*

Theorem 2 *[2] Let $\alpha \in (0,1]$ and assume f,g to be α-differentiable at a point $t > 0$. Then,*

(i) $(af+bg)^{(\alpha)} = af^{(\alpha)} + bg^{(\alpha)}$, *for all $a,b \in \mathbb{R}$;*

(ii) $(fg)^{(\alpha)} = fg^{(\alpha)} + gf^{(\alpha)}$ *;*

(iii) $(f/g)^{(\alpha)} = \dfrac{gf^{(\alpha)} - fg^{(\alpha)}}{g^2}$.

(iv) If, in addition, f is differentiable at a point $t > 0$, then

$$f^{(\alpha)}(t) = t^{1-\alpha} f'(t).$$

Additionaly, conformable fractional derivatives of certain functions as follow :

1. $(t^p)^{(\alpha)} = pt^{p-\alpha}$, for all $p \in \mathbb{R}$.
2. $(\lambda)^{(\alpha)} = 0$, for all $\lambda \in \mathbb{R}$.
3. $(e^{ct})^{(\alpha)} = ct^{1-\alpha}e^{ct}$, for all $c \in \mathbb{R}$.
4. $(e^{\frac{p}{\alpha}t^\alpha})^{(\alpha)} = pe^{\frac{p}{\alpha}t^\alpha}$, for all $p \in \mathbb{R}$.

Remark 1 *If f is differentiable at t, then f is α-differentiable at t.*

We introduce the following space :

$$C^\alpha(J,\mathbb{R}) \quad = \quad \{f : J \to \mathbb{R}, \text{ is } \alpha\text{-differentiable on } J \text{ and } f^{(\alpha)} \in C(J,\mathbb{R})\}.$$

Definition 2 (*Conformable fractional integral [2]*). *Let $\alpha \in (0,1]$ and $f : [a,\infty) \to \mathbb{R}$. The conformable fractional integral of f of order α from a to t, denoted by $I^a_\alpha(f)(t)$, is defined by*

$$I^a_\alpha(f)(t) := \int_a^t f(s)d_\alpha s := \int_a^t f(s)s^{\alpha-1}ds.$$

The considered integral is the usual improper Riemann one.

Theorem 3 *[2] If f is a continuous function in the domain of I^a_α then, for all $t \geq a$ we have*

$$\left(I^a_\alpha(f)\right)^{(\alpha)}(t) = f(t).$$

Lemma 4 *[2] Let $f : (a,b) \to \mathbb{R}$ be differentiable and $0 < \alpha \leq 1$. Then, for all $t > a$ we have*

$$I^a_\alpha(f^{(\alpha)})(t) = f(t) - f(a). \tag{3}$$

Proposition 5 *[1] Let $0 < a < b$, $f : [a,b] \to \mathbb{R}$ be continuous function and $0 < \alpha < 1$. Then for all $t \in [a,b]$ we have,*

$$|I^a_\alpha(f)(t)| \leq I^a_\alpha|f|(t).$$

3. MAIN RESULTS

In this section, we establish an existence result for the problem (1). A solution of problem (1) will be a function $x \in C^{\alpha}(J,\mathbb{R})$ for which (1) is satisfied. We introduce the notion of lower and upper solutions for the problem (1).

Definition 3 Let $v \in C^{\alpha}(J,\mathbb{R})$. We say that v is a lower solution of the boundary value problem (1), if

(i) $v^{(\alpha)}(t) \leq f(t,v(t)), \quad$ for all $t \in J$;

(ii) $v(a) \leq v(b)$.

Let $w \in C^{\alpha}(J,\mathbb{R})$. We say that w is an upper solution of the boundary value problem (1), if

(i) $w^{(\alpha)}(t) \geq f(t,w(t)), \quad$ for all $t \in J$;

(ii) $w(a) \geq w(b)$.

we define the sector

$$[v,w] = \{x \in C(J,\mathbb{R}) : v(t) \leq x(t) \leq w(t), \text{ for all } t \in J\}.$$

We consider the following problem.

$$\begin{cases} x^{(\alpha)}(t) + \alpha x(t) = f(t,\bar{x}(t)) + \alpha \bar{x}(t), \quad t \in J, \\ x(a) = x(b). \end{cases} \tag{4}$$

where

$$\bar{x}(t) = \begin{cases} v(t), & \text{if } x(t) < v(t), \\ x(t), & \text{if } v(t) \leq x(t) \leq w(t), \\ w(t), & \text{if } x(t) > w(t), \end{cases} \tag{5}$$

and $v(t) \leq w(t)$ on J.

We need the following auxiliary proposition.

Proposition 6 *For every $g \in C^{\alpha}(J,\mathbb{R})$, the problem*

$$\begin{cases} x^{(\alpha)}(t) + \alpha x(t) = g(t), \quad t \in J = [a,b], \quad a > 0, \quad 0 < \alpha < 1, \\ x(a) = x(b). \end{cases} \tag{6}$$

has a unique solution $x \in C^{\alpha}(J,\mathbb{R})$ given by :

$$x(t) := e^{-t^{\alpha}} \left(\frac{1}{e^{b^{\alpha}-a^{\alpha}} - 1} \int_a^b s^{\alpha-1} e^{s^{\alpha}} g(s)ds + \int_a^t s^{\alpha-1} e^{s^{\alpha}} g(s)ds \right) \tag{7}$$

Let us define the operators $\mathcal{N} : C(J,\mathbb{R}) \to C(J,\mathbb{R})$ by

$$\begin{aligned} \mathcal{N}(x)(t) = e^{-t^{\alpha}} \Bigg(& \frac{1}{e^{b^{\alpha}-a^{\alpha}} - 1} \int_a^b s^{\alpha-1} e^{s^{\alpha}} \Big(f(s,\bar{x}(s)) + \alpha \bar{x}(s) \Big) ds \\ & + \int_a^t s^{\alpha-1} e^{s^{\alpha}} \Big(f(s,\bar{x}(s)) + \alpha \bar{x}(s) \Big) ds \Bigg). \end{aligned}$$

Proposition 7 *If $v,w \in C^{\alpha}(J,\mathbb{R})$ are, respectively, lower and upper solutions to* (1)*, such that $v(t) \leq w(t)$, then the operator $\mathcal{N}$ is compact.*

Now, we can obtain the main theorem of this section.

Theorem 8 *If $v,w \in C^{\alpha}(J,\mathbb{R})$ are, respectively, lower and upper solutions to (1), such that $v(t) \leq w(t)$, then the problem (1) has a solution $x \in C^{\alpha}(J,\mathbb{R})$, such that $v(t) \leq x(t) \leq w(t)$ for all $t \in J$.*

Example 1 Consider the periodic problem :

$$\begin{cases} x^{(1/3)}(t) = \dfrac{2t-1-x^5(t)}{\sqrt{t}} \quad t \in J = [1/2,1], \\ \\ x(1/2) = x(1). \end{cases} \tag{8}$$

Here $n = 1$, $\alpha = 1/3$, and $f(t,x) = \dfrac{2t-1-x^5(t)}{\sqrt{t}}$. It is clear that f is a continuous function. Observe that $v = -1$ and $w = 1$ are, respectively, lower and upper solutions of (11) follows from the fact that

$$v^{(\frac{1}{3})}(t) = 0 \leq f(t,v(t)) = 2\sqrt{t},\ t \in J, \quad v(1/2) \leq v(1),$$

and

$$w^{(\frac{1}{3})}(t) = 0 \geq f(t,w(t)) = \frac{2(t-1)}{\sqrt{t}},\ t \in J, \quad w(1/2) \geq w(1),$$

Theorem 8, implies that problem (11) has a solution $x \in c^{\frac{1}{3}}([1/2,1])$, such that $-1 \leq x(t) \leq 1$, for all $t \in [1/2,1]$.

4. CONCLUSION

In this paper, we present existence of solutions for the conformable fractional differential equations with periodic boundary condition. These results are obtained by using the notion of upper and lower solutions and Schauder's fixed point theorem.

5. REFERENCES

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