VANISHING VISCOSITY FOR THE NAVIER-STOKES BOUSSINESQ SYSTEM

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ABSTRACT

This paper deals with the local well-posedness in time for the Navier-Stokes Boussinesq equations in two dimensions in the framework of a smooth vortex patch. Furthermore, we provide the inviscid limit for the velocity and the density.

1. INTRODUCTION

The system of the Navier-Stokes Boussinseq with viscosity $\mu > 0$ given by the coupled equation,

$$\begin{cases} \partial_t v_\mu + v_\mu \cdot \nabla v_\mu - \mu \Delta v_\mu + \nabla p = \theta_\mu \vec{e}_2 & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \partial_t \theta_\mu + v_\mu \cdot \nabla \theta_\mu = 0 & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \mathrm{div} v_\mu = 0, \\ (v_\mu, \theta_\mu)_{|t=0} = (v_\mu^0, {\theta_\mu}^0). \end{cases} \tag{NSB$_\mu$}$$

where $v = (v^1, v^2)\mathbb{R}^2$ refers to the velocity vector filed located in position $x \in \mathbb{R}^2$ at a time t which assumed to be incompressible, the scalar function $\theta(t,x) \in \mathbb{R}_+$ denotes the temperature or the density, $p(t,x) \in \mathbb{R}$ is the pressure which relates v and θ through an elliptic equation. $\theta_\mu \vec{e}_2$ is the buoyancy force in the direction $\vec{e}_2 = (0,1)$.

Taking the curl operator to the momentum equation in (NSB$_\mu$) we get

$$\begin{cases} \partial_t \omega_\mu + v_\mu \cdot \nabla \omega_\mu - \mu \Delta \omega_\mu = \partial_1 \theta_\mu, \\ \partial_t \theta_\mu + v_\mu \cdot \nabla \theta_\mu = 0, \\ (\theta_\mu, \omega_\mu)_{|t=0} = (\theta_\mu^0, \omega_\mu^0). \end{cases} \tag{VD$_\mu$}$$

Our goal is to prove that the system (NSB$_\mu$) is locally well-posed whenever the initial vorticity is a smooth vortex patch, that is $\omega_\mu^0 = 1_{\Omega^0}$, with the boundary $\partial\Omega_0$ is a Jordan curve with $C^{\varepsilon+1}$ regularity, $0 < \varepsilon < 1$. In addition, we prove the local persistence of geometric structures as follows, equivalently the image $\Omega_t = \Psi_\mu(t, \Omega^0)$ keeps its initial regularity, with Ψ_μ is the flow generated by the velocity v_μ,

$$\begin{cases} \partial_t \Psi_\mu(t,x) = v(t, \Psi_\mu(t,x)), \\ \Psi_\mu(0,x) = x. \end{cases}$$

Our second task is to study the inviscid limi of the system (NSB$_\mu$) towards the system (EB) given by the so-called Euler Boussinseq

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = \theta \vec{e}_2 & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \partial_t \theta + v \cdot \nabla \theta = 0 & \text{if } (t,x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\ \mathrm{div} v = 0, \\ (v, \theta)_{|t=0} = (v^0, \theta^0). \end{cases} \tag{EB}$$

and we evaluate the rate of convergence between velocities, densities and vortices when the viscosity.

Let us briefly mention some results related to the classical Euler equation. J.Y. Chemin [11] showed that if the initial boundary $\partial\Omega_0$ is $C^{1+\varepsilon}$, for $\varepsilon \in (0,1)$ then $\partial\Omega_t$ is still of class $C^{1+\varepsilon}$. In this context, Hmidi and Zerguine [25] extended the result of [11] to the stratified Euler equation. For other connected subjects in different situations we refer the reader to [1, 12, 14, 13, 16, 19, 20, 35] and the references therein. On the other hand Meddour and Zerguine [35] explored the inviscid limit of the Navier-stokes Boussinesq system to the Stratifed Euler equation in the vortex patch setting. Inspired by the works [11, 25, 35], we are mainly interested by studying the Navier-Stokes Boussinesq equations. The first result deals with the local existence and the local persistence of geometric structures of the system NSB_μ.

In particular, we have the following Theorem.

Theorem 1.1 *Let $0<\varepsilon<1, a\in(1,\infty)$ and X_0 be a family of admissible vector fields and v^0_μ be a free-divergence vector field in the sense that $\omega^0_\mu \in L^a\cap C^\varepsilon(X_0)$. Let $\theta^0_\mu \in L^2\cap C^{\varepsilon+1}(X_0)$ with $\nabla\theta^0_\mu \in L^a$, then for $\mu\in]0,1[$ the system (NSB_μ) admits a unique global solution*

$$(v_\mu,\theta_\mu)\in L^\infty\big([0,T];\mathrm{Lip}\big)\times L^\infty\big([0,T];\mathrm{Lip}\cap L^2\big).$$

and

$$\omega_\mu \in L^\infty\big([0,T];L^a\cap L^\infty\big).$$

More precisely,

$$\|\nabla v_\mu\|_{L^\infty_t L^\infty} \le C_0 e^{C_0 t}.$$

Furthermore,

$$\|\omega_\mu\|_{L^\infty_t C^\varepsilon(X_t)} + \widetilde{\|} X_\lambda\|_{L^\infty_t C^\varepsilon(X_t)} + \|\Psi_\mu\|_{L^\infty_t C^\varepsilon(X_t)} \le C_0 e^{\exp C_0 t}.$$

The second result of this paper deals with the inviscid limit for the systm (NSB_μ) to the stratified-Euler system. More precisely, we have

Theorem 1.2 *Let (v_μ,θ_μ), (v,θ), be the solution of the (NSB_μ), (EB) respectively with the same initial data satisfies the condition of Theorem 1.1 such that $\omega^0_\mu=\omega^0=\mathbf{1}_{\Omega_0}$ where Ω_0 is simply connected bounded domain. Then for all $t\ge 0, \mu\in]0,1[$, we have*

$$\|v_\mu(t)-v(t)\|_{L^2}+\|\theta_\mu(t)-\theta(t)\|_{L^2}\le C_0(\mu t)^{\frac{1}{2}}.$$

2. TOOL BOX

2.1. Function spaces

Let$(\chi,\varphi)\in\mathscr{D}(\mathbb{R}^2)\times\mathscr{D}(\mathbb{R}^2)$ be a radial cut-off functions be such that supp $\chi\subset\{\xi\in\mathbb{R}^2 : \|\xi\|\le 1\}$ and supp $\varphi(\xi)\subset\{\xi\in\mathbb{R}^2 : 1/2\le\|\xi\|\le 2\}$, so that

$$\chi(\xi)+\sum_{q\ge 0}\varphi(2^{-q}\xi)=1.$$

Through χ and φ, the Littlewood-Paley or frequency cut-off operators $(\Delta_q)_{q\ge -1}$ and $(\dot\Delta_q)_{q\ge -1}$ are defined for $u\in\mathscr{S}'(\mathbb{R}^2)$

$$\Delta_{-1}u=\chi(\mathrm{D})u,\ \Delta_q u=\varphi(2^{-q}\mathrm{D})u \ \text{ for } \ q\in\mathbb{N},\quad \dot\Delta_q u=\varphi(2^{-q}\mathrm{D})u \ \text{ for } \ q\in\mathbb{Z}.$$

where in general case $f(\mathrm{D})$ stands the pseudo-differential operator $u\mapsto\mathscr{F}^{-1}(f\mathscr{F}u)$ with constant symbol. The lower frequencies sequence $(S_q)_{q\ge 0}$ is defined for $q\ge 0$,

$$S_q u \triangleq \sum_{j\le q-1}\Delta_j u.$$

In accordance of the previous properties we derive the well-known decomposition of unity

$$u = \sum_{q\geq -1} \Delta_q u, \quad u = \sum_{q\in\mathbb{Z}} \dot{\Delta}_q u.$$

The results currently available allow us to define the inohomogeneous Besov denoted $B^s_{p,r}$ (resp. $\dot{B}^s_{p,r}$) and defined in the following way.

Definition 2.1 *For $(p,r,s) \in [1,+\infty]^2 \times \mathbb{R}$, the inhomogeneous Besov spaces $B^s_{p,r}$ (resp. homogeneous Besov spaces $\dot{B}^s_{p,r}$) are defined by*

$$B^s_{p,r} = \{u \in \mathscr{S}'(\mathbb{R}^2) : \|u\|_{B^s_{p,r}} < +\infty\}, \quad \dot{B}^s_{p,r} = \{u \in \mathscr{S}'(\mathbb{R}^2)_{|\mathbb{P}} : \|u\|_{\dot{B}^s_{p,r}} < +\infty\},$$

where $\mathbb{P}$ refers to the set of polynomial functions in $\mathbb{R}^2$ so that

$$\|u\|_{B^s_{p,r}} \triangleq \begin{cases} \left(\sum_{q\geq -1} 2^{rqs}\|\Delta_q u\|^r_{L^p}\right)^{1/r} & \text{if } r \in [1,+\infty[, \\ \sup_{q\geq -1} 2^{qs}\|\Delta_q u\|_{L^p} & \text{if } r = +\infty. \end{cases}$$

and

$$\|u\|_{\dot{B}^s_{p,r}} \triangleq \begin{cases} \left(\sum_{q\in\mathbb{Z}} 2^{rqs}\|\dot{\Delta}_q u\|^r_{L^p}\right)^{1/r} & \text{if } r \in [1,+\infty[, \\ \sup_{q\in\mathbb{Z}} 2^{qs}\|\dot{\Delta}_q u\|_{L^p} & \text{if } r = +\infty. \end{cases}$$

The Bernstein's inequalities are listed in the following lemma.

Lemma 2.2 *There exists a constant $C > 0$ such that for $1 \leq a \leq b \leq \infty$, for every function u and every $q \in \mathbb{N} \cup \{-1\}$, we have*

- **(i)** $\sup_{|\alpha|=k} \|\partial^\alpha S_q u\|_{L^b} \leq C^k 2^{q(k+2(1/a-1/b))}\|S_q u\|_{L^a}$.
- **(ii)** $C^{-k}2^{qk}\|\Delta_q u\|_{L^a} \leq \sup_{|\alpha|=k} \|\partial^\alpha \Delta_q u\|_{L^a} \leq C^k 2^{qk}\|\Delta_q u\|_{L^a}$.

As a consequence of Bernstein inequality, we have

Proposition 2.3 *For $(s,\widetilde{s},p,p_1,p_2,r_1,r_2) \in \mathbb{R}^2\times]1,\infty[\times[1,\infty]^4$ with $\widetilde{s} \leq s$, $p_1 \leq p_2$ and $r_1 \leq r_2$, then we have*

- **(i)** $B^s_{p,r} \hookrightarrow B^{\widetilde{s}}_{p,r}$.
- **(ii)** $B^s_{p_1,r_1}(\mathbb{R}^2) \hookrightarrow B^{s+2(1/p_2-1/p_1)}_{p_2,r_2}(\mathbb{R}^2)$.

Now, we state Bony's decomposition [8] which allows us to split formally the product of two tempered distributions u and v into three pieces. More precisely, we have.

Definition 2.4 *For a given $u, v \in \mathscr{S}'$ we have*

$$uv = T_u v + T_v u + \mathscr{R}(u,v),$$

with

$$T_u v = \sum_q S_{q-1} u \Delta_q v, \quad \mathscr{R}(u,v) = \sum_q \Delta_q u \widetilde{\Delta}_q v \quad \text{and} \quad \widetilde{\Delta}_q = \Delta_{q-1} + \Delta_q + \Delta_{q+1}.$$

The mixed space-time spaces are stated as follows.

Definition 2.5 *Let $T > 0$ and $(s,\beta,p,r) \in \mathbb{R}\times[1,\infty]^3$. We define the spaces $L^\beta_T B^s_{p,r}$ and $\widetilde{L}^\beta_T B^s_{p,r}$ respectively by :*

$$L^\beta_T B^s_{p,r} \triangleq \left\{u : [0,T] \to \mathscr{S}'; \|u\|_{L^\beta_T B^s_{p,r}} = \left\|\left(2^{qs}\|\Delta_q u\|_{L^p}\right)_{\ell^r}\right\|_{L^\beta_T} < \infty\right\},$$

$$\widetilde{L}^\beta_T B^s_{p,r} \triangleq \left\{u : [0,T] \to \mathscr{S}'; \|u\|_{\widetilde{L}^\beta_T B^s_{p,r}} = \left(2^{qs}\|\Delta_q u\|_{L^\beta_T L^p}\right)_{\ell^r} < \infty\right\}.$$

The following result is a useful result in our approach. For the proof see [19, Corollary 1]

Corollary 2.6 *Given $\varepsilon \in]0,1[$ and X be a vector field be such that $X, \mathrm{div} X \in C^{\varepsilon}$. Then for f be a Lipschitz scalar function $k \in \{1,2\}$ the following statement holds.*

$$\|(\partial_k X)\cdot\nabla f\|_{C^{\varepsilon-1}} \leq C\|\nabla f\|_{L^\infty}\big(\|\mathrm{div}X\|_{C^\varepsilon}+\|X\|_{C^\varepsilon}\big).$$

2.2. particle results

In this subsection, we give some preparatory results freely used throughout our analysis.

$$\begin{cases} \partial_t a + v\cdot\nabla a - \mu\Delta a = g, \\ a_{|t=0} = a^0. \end{cases} \tag{1}$$

We start with the persistence of Besov regularity for (1) whose proof may be found in [5]

Proposition 2.7 *Let $(s,r,p) \in]-1,1[\times[1,\infty]^2$ and v be a smooth vector field in free-divergence. Assume that $(a^0,g) \in B^s_{p,r}\times L^1_{\mathrm{loc}}(\mathbb{R}_+;B^s_{p,r})$. Then for every smooth solution a of (1) and $t\geq 0$ we have*

$$\|a(t)\|_{B^s_{p,r}} \leq Ce^{CV(t)}\left(\|a^0\|_{B^s_{p,r}}+\int_0^t e^{-CV(\tau)}\|g(\tau)\|_{B^s_{p,r}}d\tau\right),$$

with the notation

$$V(t)=\int_0^t\|\nabla v(\tau)\|_{L^\infty}d\tau,$$

where $C=C(s)$ being a positive constant.

The statement of maximal regularity for (1) in mixed space-time Besov space is given by the following result. For the proof see [5].

Proposition 2.8 *Let $(s,p_1,p_2,r) \in]-1,1[\times[1,\infty]^3$ and v be a free-divergence vector field belongs to $L^1_{loc}(\mathbb{R}_+;\mathrm{Lip})$ then there exists a constant $C\geq 0$, so that for every smooth solution a of (1) we have for all $t\geq 0$*

$$\mu^{\frac{1}{r}}\|a\|_{\widetilde{L}^r_t B^{s+\frac{2}{r}}_{p_1,p_2}} \leq Ce^{CV(t)}(1+\mu t)^{\frac{1}{r}}\left(\|a^0\|_{B^s_{p_1,p_2}}+\|g\|_{L^1_t B^s_{p_1,p_2}}\right).$$

Next, we have the classical Cálderon Zygmund inequality.

Proposition 2.9 *Let $p\in]1,+\infty[$ and v be a free-divergence vector field whose vorticity $\omega \in L^p$. Then $\nabla v \in L^p$ and*

$$\|\nabla v\|_{L^p} \leq C\frac{p^2}{p-1}\|\omega\|_{L^p}.$$

with C being a universal constant.

At this stage, we define the anisotropic Hölder spaces as follows

Definition 2.10 *Let $\varepsilon \in]0,1[$. A family of vector fields $X=(X_\lambda)_{\lambda\in\Lambda}$ is said to be admissible if and only if the following assertions are hold.*

- **(i)** *Regularity : $\forall\lambda\in\Lambda \quad X_\lambda, \mathrm{div}X_\lambda \in C^\varepsilon$.*
- **(ii)** *Non-degeneray : $I(X) \triangleq \inf_{x\in\mathbb{R}^N}\sup_{\lambda\in\Lambda} \mid X_\lambda(x) \mid > 0$.*

Setting

$$\widetilde{\|}X_\lambda \| C^\varepsilon \triangleq \|X_\lambda\|_{C^\varepsilon} + \|\mathrm{div} X_\lambda\|_{C^\varepsilon}.$$

Definition 2.11 *Let $X = (X_\lambda)_{\lambda\in\Lambda}$ be an admissible family. The action of each factor X_λ on $u \in L^\infty$ is defined as the directional derivative of u along X_λ by the formula,*

$$\partial_{X_\lambda} u = \mathrm{div}(uX_\lambda) - u\mathrm{div} X_\lambda.$$

The concept of anisotropic Hölder space, will be noted by $C^\varepsilon(X)$ is defined below.

Definition 2.12 *Let $\varepsilon \in]0,1[$ and X be an admissible family of vector fields. We say that $u \in C^\varepsilon(X)$ if and only if :*

(i) *$u \in L^\infty$ and satisfies*

$$\forall \lambda \in \Lambda, \partial_{X_\lambda} u \in C^{\varepsilon-1}, \quad \sup_{\lambda\in\Lambda} \|\partial_{X_\lambda} u\|_{C^{\varepsilon-1}} < +\infty.$$

(ii) *$C^\varepsilon(X)$ is a normed space with*

$$\|u\|_{C^\varepsilon(X)} \triangleq \frac{1}{I(X)} \left(\|u\|_{L^\infty} \sup_{\lambda\in\Lambda} \widetilde{\|}X_\lambda\|_{C^\varepsilon} + \sup_{\lambda\in\Lambda} \|\partial_{X_\lambda} u\|_{C^{\varepsilon-1}} \right).$$

The next result play a major role in the proof of our main results. We refer the reader to [11].

Theorem 2.13 *Let $\varepsilon \in]0,1[$ and $X = (X_{t,\lambda})_{\lambda\in\Lambda}$ be a family of vector fields as in Definition 2.12. Let v be a free-divergence vector field such that its vorticity ω belongs to $L^2 \cap C^\varepsilon(X)$. Then there exists a constant C depending only on ε, such that*

$$\|\nabla v\|_{L^\infty} \leq C \left(\|\omega\|_{L^2} + \|\omega\|_{L^\infty} \log\left(e + \frac{\|\omega\|_{C^\varepsilon(X)}}{\|\omega\|_{L^\infty}} \right) \right). \tag{2}$$

According Danchin's result [12], the class C^ε_Σ doesn't covers only the vortex patch of the type $\omega_0 = \mathbf{1}_{\Omega_0}$, but also encompass the so-called general vortex. Specifically, we have.

Proposition 2.14 *Let Ω_0 be a $C^{1+\varepsilon}-$bounded domain, with $0 < \varepsilon < 1$. Then for every function $f \in C^\varepsilon$, we have*

$$f\mathbf{1}_{\Omega_0} \in C^\varepsilon_\Sigma.$$

2.3. A priori estimates

In this part we shall give some a priori estimates for the velocity and the vorticity.

Proposition 2.15 *Let v_μ be a smooth divergence-free vector field and θ_μ be a smooth solution of the equation (**??**). Then the following assertions are hold.*

(i) *For $p \in [1,\infty]$ and $t \geq 0$ we have*

$$\|\theta_\mu(t)\|_{L^p} \leq \|\theta^0_\mu\|_{L^p}.$$

(ii) *For $p \in [1,\infty]$ and $t \geq 0$ we have*

$$\|\nabla\theta_\mu(t)\|_{L^p} \leq \|\nabla\theta^0_\mu\|_{L^p} e^{CV_\mu(t)}.$$

with $V_\mu(t) = \int_0^t \|\nabla v_\mu(\tau)\|_{L^\infty} d\tau$.

(iii) *For $p \in [1,\infty]$ and $t \geq 0$ we have*

$$\|\omega_\mu(t)\|_{L^p} \leq C_0 e^{CV_\mu(t)}.$$

with $V(t) = \int_0^t \|\nabla v_\mu(\tau)\|_{L^\infty} d\tau.$

Proof. (i) According to (1) we can express the density $\theta_\mu(t)$ by the initial value θ_μ^0 and the flow Ψ as follows

$$\theta_\mu(t,x) = \theta_\mu^0(\Psi^{-1}(t,x)).$$

Taking the L^p-norm to this equation and thanks to incompressible condition we infer that

$$\|\theta_\mu(t)\|_{L^p} \leq \|\theta_\mu^0\|_{L^p}. \tag{3}$$

.

(ii) Taking the partial derivative ∂_j to $\theta_\mu-$ equation to obtain

$$\partial_t \partial_j \theta + v \cdot \nabla\theta = -\partial_j\theta \cdot \nabla v,$$

The L^p-estimate for the above gives

$$\|\nabla\theta(t)\|_{L^p} \lesssim \|\nabla\theta_0\|_{L^p} + \int_0^t \|\nabla\theta(\tau)\|_{L^p}\|\nabla v(\tau)\|_{L^\infty} d\tau.$$

The Gronwall's inequality implies that

$$\|\nabla\theta(t)\|_{L^p} \leq \|\nabla\theta_0\|_{L^p} e^{CV(t)}.$$

(iii) The L^p-estimate for the ω_μ equation gives

$$\|\omega_\mu(t)\|_{L^p} \lesssim \|\omega_\mu^0\|_{L^p} + \int_0^t \|\nabla\theta(\tau)\|_{L^p}\|\nabla v(\tau)\|_{L^\infty} d\tau.$$

Combining the last two estimates we find the desired result . ■

Proof of Theorem 1.1 The existence part of the theorem is classical and can be done for example by using a standard recursive method, see, e.g. [19]. We will control the quantities $\|\nabla v(t)\|_{L^\infty}$ and $\|\omega_\mu(t)\|_{C^\varepsilon(X)}$ for every $t \geq 0$. For this aim, appalling the operator $\partial_{X_t,\lambda}$ to ω_μ equation, we have

$$(\partial_t + v\cdot\nabla - \mu\Delta)\partial_{X_t,\lambda}\omega_\mu = X_{t,\lambda}\cdot\nabla\partial_1\theta_\mu - \mu[\Delta, X_{t,\lambda}]\omega_\mu.$$

For commutator $\mu[\Delta, X_{t,\lambda}]\omega_\mu$. From Bony's decomposition, we write

$$\mu[\Delta, X_{t,\lambda}]\omega_\mu = \mathfrak{A} + \mu\mathfrak{B},$$

with

$$\mathfrak{A} \triangleq 2\mu T_{\nabla X^i_{t,\lambda}}\partial_i\nabla\omega_\mu + 2\mu T_{\partial_i\nabla\omega_\mu}\nabla X^i_{t,\lambda} + \mu T_{\Delta X^i_{t,\lambda}}\partial_i\omega_\mu + \mu T_{\partial_i\omega_\mu}\Delta X^i_{t,\lambda}.$$

and

$$\mathfrak{B} \triangleq 2\mathscr{R}(\nabla X^i_{t,\lambda}, \partial_i\nabla\omega_\mu) + \mathscr{R}(\Delta X^i_{t,\lambda}, \partial_i\omega_\mu).$$

According to Theorem 3.38 page 162 in [5] , we have

$$\|\partial_{X_\lambda}\omega_\mu\|_{L^\infty_t C^{\varepsilon-1}} \leq Ce^{CV_\mu(t)}\Big(\|\partial_{X_0,\lambda}\omega_\mu^0\|_{C^{\varepsilon-1}} + (1+\mu t)\|\mathfrak{A}\|_{L^\infty_t C^{\varepsilon-3}} + \mu\|\mathfrak{B}\|_{\widetilde{L}^1_t C^{\varepsilon-1}} + \|\partial_{X_\lambda}\partial_1\theta_\mu\|_{L^1_t C^{\varepsilon-1}}\Big). \tag{4}$$

From [5, 20] we have the following estimate

$$\|\mathfrak{A}\|_{L^\infty_t C^{\varepsilon-3}} \leq C\|\omega\|_{L^\infty_t L^\infty}\|X_\lambda\|_{L^\infty_t C^\varepsilon}.$$

Thanks to Proposition 2.15, we get

$$\|\mathfrak{A}\|_{L_t^\infty C^{\varepsilon-3}} \leq C_0 e^{CV_\mu(t)} \|X_\lambda\|_{L_t^\infty C^\varepsilon}. \tag{5}$$

Again from [5, 20] we have the following estimate

$$\|\mathfrak{B}\|_{\widetilde{L}_t^1 C^{\varepsilon-1}} \leq C\|\omega\|_{\widetilde{L}_t^1 B^2_{\infty,\infty}} \|X_\lambda\|_{L_t^\infty C^\varepsilon}. \tag{6}$$

For the term $\|\omega_\mu\|_{\widetilde{L}_t^1 B^2_{\infty,\infty}}$, we use the Proposition 2.8 for $a = \omega_\mu, g = \partial_1\theta_\mu, r = 1, s = 0$, and $p_1 = p_2 = \infty$, we obtain

$$\mu\|\omega_\mu\|_{\widetilde{L}_t^1 B^2_{\infty,\infty}} \leq Ce^{CV_\mu(t)}(1+\mu t)\Big(\|\omega_\mu^0\|_{B^0_{\infty,\infty}} + \int_0^t \|\partial_1\theta_\mu(\tau)\|_{B^0_{\infty,\infty}} d\tau\Big).$$

The embedding $L^\infty \hookrightarrow B^0_{\infty,\infty}$ implies that

$$\mu\|\omega_\mu\|_{\widetilde{L}_t^1 B^2_{\infty,\infty}} \leq Ce^{CV_\mu(t)}(1+\mu t)\big(\|\omega_\mu^0\|_{L^\infty} + \|\nabla\theta_\mu\|_{L_t^1 L^\infty}\big).$$

Using Proposition 2.8 we obtain

$$\mu\|\omega_\mu\|_{\widetilde{L}_t^1 B^2_{\infty,\infty}} \leq C_0 e^{CV_\mu(t)}(1+\mu t)(1+t)$$

Plugging the last estimate into (7), we infer that

$$\|\mathfrak{B}\|_{\widetilde{L}_t^1 C^{\varepsilon-1}} \leq C_0 e^{CV_\mu(t)}(1+\mu t)(1+t)\|X_\lambda\|_{L_t^\infty C^\varepsilon}. \tag{7}$$

To treat the quantity $\|\partial_{X_\lambda}\partial_1\theta_\mu\|_{L_t^1 C^{\varepsilon-1}}$ we note that

$$\partial_{X_{\tau,\lambda}}\partial_1\theta_\mu = \partial_1(\partial_{X_{\tau,\lambda}}\theta_\mu) - \partial_{\partial_1 X_{\tau,\lambda}}\theta_\mu. \tag{8}$$

It follows , from taking the $C^{\varepsilon-1}-$ norm to this equation

$$\|\partial_{X_{\tau,\lambda}}\partial_1\theta_\mu(\tau)\|_{C^{\varepsilon-1}} \lesssim \|\partial_1(\partial_{X_{\tau,\lambda}}\theta_\mu)(\tau)\|_{C^{\varepsilon-1}} + \|(\partial_1 X_{\tau,\lambda})\cdot\nabla\theta_\mu(\tau)\|_{C^{\varepsilon-1}}.$$

Moreover using the fact $\partial_1 : C^\varepsilon \longrightarrow C^{\varepsilon-1}$ is a continuous map and Corollary 2.6, we get

$$\|\partial_{X_{\tau,\lambda}}\partial_1\theta_\mu(\tau)\|_{C^{\varepsilon-1}} \lesssim \|\partial_{X_{\tau,\lambda}}\theta_\mu(\tau)\|_{C^\varepsilon} + \|\nabla\theta_\mu(\tau)\|_{L^\infty}\widetilde{\|}X_{\tau,\lambda}\|_{C^\varepsilon} e^{CV_\mu(t)}.$$

From Proposition 2.7 and Proposition 2.15, we find

$$\|\partial_{X_{\tau,\lambda}}\partial_1\theta_\mu(\tau)\|_{L_t^1 C^{\varepsilon-1}} \leq \|\partial_{X(0)_\lambda}\theta_\mu^0\|_{C^\varepsilon} e^{CV_\mu(t)} t + C_0\widetilde{\|}X_{\tau,\lambda}\|_{C^\varepsilon} e^{CV_\mu(t)} \tag{9}$$

Summing (5),(7),(9) and punting them in (4),such that $\mu \in]0,1[$ we infer that

$$\|\partial_{X_{t,\lambda}}\omega_\mu\|_{L_t^\infty C^{\varepsilon-1}} \leq C_0 e^{CV_\mu(t)}(1+t)^2\widetilde{\|}X_{\lambda,t}\|_{L^\infty C^\varepsilon}. \tag{10}$$

On other hand bound by using Proposition 2.7, we find

$$\|X_{t,\lambda}\|_{C^\varepsilon} \leq Ce^{CV_\mu(t)}\Big(\|X_{0,\lambda}\|_{C^\varepsilon} + \int_0^t e^{-CV_\mu(\tau)}\|\partial_{X_{\tau,\lambda}} v_\mu(\tau)\|_{C^\varepsilon} d\tau\Big). \tag{11}$$

We use the following result which its proof can be found in [5, 11]

$$\|\partial_{X_{t,\lambda}} v_\mu(t)\|_{C^\varepsilon} \leq C\big(\|\nabla v_\mu(t)\|_{L^\infty}\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} + \|\omega_\mu(t)\|_{C^{\varepsilon-1}}\big).$$

Thanks to (10), we find

$$\|\partial_{X_{t,\lambda}} v_\mu(t)\|_{C^\varepsilon} \leq C\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon}\Big(\|\nabla v_\mu(t)\|_{L^\infty} + C_0 e^{CV_\mu(t)}(1+t)^2\Big). \tag{12}$$

Plunging (12) in(11), we get

$$\|X_{t,\lambda}\|_{C^\varepsilon} \leq Ce^{CV_\mu(t)}\left(\|X_{0,\lambda}\|_{C^\varepsilon} + C_0\int_0^t e^{-CV_\mu(\tau)}\widetilde{\|}X_{\tau,\lambda}\|_{C^\varepsilon}\Big(\|\nabla v_\mu(\tau)\|_{L^\infty} + (1+\tau)^2\Big)d\tau\right).$$

For the term $\|\mathrm{div}X_{t,\lambda}\|_{C^\varepsilon}$ by applying Proposition 2.7 to $(\partial_t + v\cdot\nabla)\mathrm{div}X_{t,\lambda} = 0$, we get

$$\|\mathrm{div}X_{t,\lambda}\|_{C^\varepsilon} \leq Ce^{CV_\mu(t)}\|\mathrm{div}X_{0,\lambda}\|_{C^\varepsilon}. \tag{13}$$

Combining the last two estimates we have

$$e^{-CV_\mu(t)}\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} \leq C\left(\widetilde{\|}X_{0,\lambda}\|_{C^\varepsilon} + C_0\int_0^t e^{-CV_\mu(\tau)}\widetilde{\|}X_{\tau,\lambda}\|_{C^\varepsilon}\Big(\|\nabla v_\mu(\tau)\|_{L^\infty} + (1+\tau)^2\Big)d\tau\right). \tag{14}$$

The Gronwall's inequality gives

$$\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} \leq C_0 e^{C_0V_\mu(t)}e^{C_0t^3}. \tag{15}$$

Gathering (10) and (15), one has

$$\|\partial_{X_{t,\lambda}}\omega_\mu(t)\|_{C^{\varepsilon-1}} \leq C_0 e^{C_0V_\mu(t)}e^{C_0t^3}.$$

Moreover, from the last two estimates and Propitiation 2.15, we get

$$\|\partial_{X_{t,\lambda}}\omega_\mu(t)\|_{C^{\varepsilon-1}} + \|\omega_\mu(t)\|_{L^\infty}\widetilde{\|}X_{t,\lambda}\|_{C^\varepsilon} \leq C_0 e^{C_0V_\mu(t)}e^{C_0t^3}. \tag{16}$$

Now, we recall that

$$\|\omega\|_{C^\varepsilon(X)} \triangleq \frac{1}{I(X_t)}\left(\|\omega\|_{L^\infty}\sup_{\lambda\in\Lambda}\widetilde{\|}X_\lambda\|_{C^\varepsilon} + \sup_{\lambda\in\Lambda}\|\partial_{X_\lambda}\omega\|_{C^{\varepsilon-1}}\right). \tag{17}$$

To control the term $I(X_t)$ we apply the derivative in time to the quantitity $\partial_{X_{0,\lambda}}\Psi$, it follows

$$\begin{cases} \partial_t\partial_{X_{0,\lambda}}\Psi(t,x) = \nabla v(t,\Psi(t,x))\partial_{X_{0,\lambda}}\psi(t,x) \\ \partial_{X_{0,\lambda}}\Psi(0,x) = X_{0,\lambda}. \end{cases}$$

The time reversibilty of the previous equation and Gronwall's inequality ensure that

$$|X_{0,\lambda}(x)| \leq |\partial_{X_{0,\lambda}}\Psi(t,x)|e^{V_\mu(t)}.$$

From **(ii)** in Definition 2.10 we get

$$I(X_t) \geq I(X_0)e^{-V_\mu(t)} > 0. \tag{18}$$

Thanks to (16), (17) and (18), we have

$$\|\omega_\mu(t)\|_{C^\varepsilon(X_t)} \leq C_0 e^{C_0t^3}e^{C_0V_\mu(t)}. \tag{19}$$

According to Theorem 2.13 and Proposition 2.15 , we obtain

$$\|\nabla v_\mu(t)\|_{L^\infty} \leq C\left(C_0 + C_0\log\left(e + \frac{\|\omega_\mu(t)\|_{C^\varepsilon(X)}}{\|\omega_\mu(t)\|_{L^\infty}}\right)\right).$$

The monotonicity of function $x \longmapsto \log(e+\frac{a}{x})$ gives

$$\|\nabla v_\mu(t)\|_{L^\infty} \le C_0\left(C_0 + C_0\log\left(e+\frac{\|\omega_\mu(t)\|_{C^\varepsilon(X)}}{\|\omega_\mu^0\|_{L^\infty}}\right)\right).$$

It follows from (19) that

$$\|\nabla v_\mu(t)\|_{L^\infty} \le C_0\Big((1+t)^3 + \int_0^t \|\nabla v_\mu(\tau)\|_{L^\infty} d\tau\Big).$$

Again, Gronwall's inequality gives

$$\|\nabla v_\mu(t)\|_{L^\infty} \le C_0 e^{C_0 t}. \tag{20}$$

Together with (19), we have

$$\|\omega_\mu(t)\|_{C^\varepsilon(X_t)} \le C_0 e^{\exp C_0 t^t}. \tag{21}$$

To control he term Ψ_μ in $C^\varepsilon(X_t)$. we recall that $\partial_{X_{0,\lambda}}\Psi_\mu(t) = X_{t,\lambda}\circ\Psi_\mu(t)$. Thus, we get

$$\|X_{t,\lambda}\circ\Psi_\mu(t)\|_{C^\varepsilon} \le \|X_{t,\lambda}\|_{C^\varepsilon}\|\nabla\Psi_\mu(t)\|_{L^\infty}^\varepsilon \le \|X_{t,\lambda}\|_{C^\varepsilon} e^{CV_\mu(t)},$$

where, we have used $\|\nabla\Psi_\mu(t)\|_{L^\infty} \le e^{CV_\mu(t)}$. Consequently,

$$\|\Psi_\mu(t)\|_{C^\varepsilon(X_t)} \le C_0 e^{\exp C_0 t}. \tag{22}$$

The proof of Theorem 1.1 is finished.

3. INVISCID LIMIT

Proof of Theorem 1.2

Taking the difference between (NSB$_\mu$) and (EB), by setting $U = v_\mu - v, \Theta = \theta_\mu - \theta$ and $P = p_\mu - p$ we find out that the triplet (U,Θ,P) gouverns the following evolution system.

$$\begin{cases} \partial_t U + v_\mu\cdot\nabla U - \mu\Delta U = \Delta v - \nabla P + \Theta\vec{e}_2 - U\cdot\nabla v, \\ \partial_t\Theta + v_\mu\cdot\nabla\Theta = -U\cdot\nabla\theta, \\ \nabla\cdot U = 0, \\ (U,\Theta)_{|t=0} = (U_0,\Theta_0). \end{cases} \tag{D$_\mu$}$$

Multiplying the first equation in the system (D$_\mu$) by U and integrating by part over $\mathbb{R}^2$, such that $\mathrm{div} v_\mu = \mathrm{div} v = 0$ and Hölder's inequality ensure that

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|_{L^2}^2 + \mu\|\nabla U(t)\|_{L^2}^2 \le \mu\|\nabla v(t)\|_{L^2}\|\nabla U(t)\|_{L^2} + \|\nabla v(t)\|_{L^\infty}\|U(t)\|_{L^2} + \|\Theta(t)\|_{L^2}\|U(t)\|_{L^2}.$$

Young inequality implies that

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|_{L^2}^2 + \mu\|\nabla U(t)\|_{L^2}^2 \le \frac{\mu}{2}\|\nabla v(t)\|_{L^2}^2 + \frac{\mu}{2}\|\nabla U(t)\|_{L^2}^2 + \Big(\|U(t)\|_{L^2}^2 + \|\Theta(t)\|_{L^2}^2\Big)\big(\|\nabla v(t)\|_{L^\infty}+1\Big).$$

Integrating in time this inequality ,we find

$$\|U(t)\|_{L^2}^2 + \frac{\mu}{2}\|\nabla U(t)\|_{L^2}^2 \le \|U^0\|_{L^2}^2 + \frac{\mu}{2}\|\nabla v\|_{L_t^2 L^2} + \int_0^t \Big(\|U(\tau)\|_{L^2}^2 + \|\Theta(\tau)\|_{L^2}^2\Big)\big(\|\nabla v(\tau)\|_{L^\infty}+1\Big)d\tau. \tag{23}$$

Similarly for $\Theta-$equation, we have

$$\frac{1}{2}\frac{d}{dt}\|\Theta(t)\|_{L^2}^2 \leq \|\nabla\theta(t)\|_{L^\infty}\|U(t)\|_{L^2}^2.$$

Integrating in time over $[0,t]$, it follows

$$\|\Theta(t)\|_{L^2}^2 \leq \|\Theta^0\|_{L^2}^2 + \int_0^t \|\nabla\theta(\tau)\|_{L^\infty}\|U(\tau)\|_{L^2}^2 d\tau. \tag{24}$$

Gathering (23) and (24), one has

$$\begin{aligned}\Pi(t) \quad &\lesssim \quad \Pi(0) + \int_0^t \Pi(\tau)\Big(1 + \|\nabla\theta(\tau)\|_{L^\infty} + \|\nabla v(\tau)\|_{L^\infty}\Big)d\tau \\ &+ \quad \mu\|\nabla v\|_{L_t^2 L^2},\end{aligned} \tag{25}$$

with $\Pi(t) = \|v_\mu(t) - v(t)\|_{L^2}^2 + \|\theta_\mu(t) - \theta(t)\|_{L^2}^2$, so Gronwall's inequality gives

$$\Pi(t) \leq Ce^{t+V_\mu(t)+V(t)+\|\nabla\theta\|_{L_t^1 L^\infty}}\Big(\Pi(0) + \mu\|\Delta v_\mu\|_{L_t^2 L^2}\Big).$$

Again, Hölder's inequality in time variable and using Proposition 2.9, we get

$$\Pi(t) \leq Ce^{t+V_\mu(t)+V(t)+\|\nabla\theta\|_{L_t^1 L^\infty}}\Big(\Pi(0) + (\mu t)\|\omega_\mu\|_{L_t^\infty L^2}\Big).$$

Thanks to Theorem and Proposition 2.15, we find that

$$\Pi(t) \leq C_0 e^{C_0 t}(\mu t).$$

Consequently,

$$\|v_\mu(t) - v(t)\|_{L^2} + \|\theta_\mu(t) - \theta(t)\|_{L^2} \leq C_0(\mu t)^{\frac{1}{2}}.$$

This completes the proof of Theorem 1.2.

4. REFERENCES

[1] H. Abidi and R. Danchin : *Optimal bounds for the inviscid limit of Navier-Stokes equations*. Asymptot. Anal., 38(1), 35–46 (2004).

[2] H. Abidi and T. Hmidi : *On the global well-posedness for Boussinesq System*. J. Diff. Equa. 233 (1), 199-220 (2007).

[3] H. Abidi and P. Zhang : *On the global well-posedness of* 2*D Boussinesq system with variable viscosity*. Adv. Math. 305, 1202–1249 (2017)

[4] J. L. Boldrini, B. Climent-Ezquerra, M. D. Rojas-Medar and M. A. Rojas-Medar : *On an iterative method for approximate solutions of a generalized Boussinesq model*. Journal of Mathematical Fluid Mechanics volume 13, 33–53 (2011).

[5] H. Bahouri, J.-Y. Chemin and R. Danchin : *Fourier analysis and nonlinear partial differential equations*. Springer-Verlag Berlin Heidelberg (2011).

[6] J. T. Beale, T. Kato and A. Majda : *Remarks on the breakdown of smooth solutions for the* $3D-$*Euler equations*. Commun. Math. Phys. 94, 61–66 (1984).

[7] A. Bertozzi and P. Constantin : *Global regularity for vortex patches*. Comm. Math. Phys. 152 (1), 19–28 (1993).

[8] J.-M. Bony : *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires.* Ann. de l'Ecole Norm. Sup. (14), 209–246 (1981).

[9] J. R. Cannon and E. Dibenedetto : *The Initial Value Problem for the Boussinesq Equations with Data in L^p*. Lecture Notes in Math. 771, Berlin-Heidelberg-New York : Springer, 129–144 (1980).

[10] D. Chae : *Global regularity for the $2D-$Boussinesq equations with partial viscous terms*. Advances in Math. 203 (2), 497–513 (2006).

[11] J.-Y. Chemin : *Perfect incompressible Fluids.* Oxford University Press (1998).

[12] R. Danchin : *Poches de tourbillon visqueuses.* Math. Pures Appl. (9) 76, 609–647 (1997).

[13] R. Danchin and M. Paicu : *Global well-posedness issues for the inviscid Boussinesq system with Yudovich's type data*. Comm. Math. Phys. 290, 1–14 (2009).

[14] R. Danchin and X. Zhang : *Global persistence of geometrical structures for the Boussinesq equation with no diffusion*. Communications in Partial Differential Equations, 42(1), 68–99 (2017).

[15] N. Depauw : *Poche de tourbillon pour Euler $2D$ dans un ouvert à bord*. J. Math. Pures Appl. (9) 78, no. 3, 313–351 (1999).

[16] F. Fanelli : *Conservation of geometric structures for non-homogeneous inviscid incompressible fluids*. Comm. Partial Differential Equations 37(9), 1553–1595 (2012).

[17] D. Francesco : *Global weak solutions for Boussinesq system with temperature dependent viscosity and bounded temperature*. Adv. Differential Equations, 21 (11-12) : 1001–1048 (2015).

[18] B. Guo : *Spectral method for solving two-dimensional Newton-Boussinesq equation*. Acta Math. Appl. Sinica 5, 27–50 (1989).

[19] Z. Hassainia and T. Hmidi : *On the inviscid Boussinesq system with rough initial data*. J. Math. Anal. Appl. 430, no. 2, 777–809 (2015).

[20] T. Hmidi : *Régularité höldérienne des poches de tourbillon visqueuses*. J. Math. Pures Appl. (9) 84, no. 11, 1455–1495 (2005).

[21] T. Y. Hou and C. Li : *Global well-Posedness of the viscous Boussinesq equations*. Discrete and Continuous Dynamical Systems. Series A, 12 (1), 1–12 (2005).

[22] T. Hmidi and S. Keraani : *On the global well-posedness of the two-dimensional Boussinesq system with a zero diffusivity*. Adv. Differential Equations 12 (4), 461-480 (2007).

[23] T. Hmidi, S. Keraani and F. Rousset : *Global well-posedness for an Euler-Boussinesq system with critical dissipation*. Commun. Part. Diff. Eqs. 36(3), 420-445, (2011).

[24] T. Hmidi and M. Zerguine : *On the global well-posedness of the Euler-Boussinesq system with fractional dissipation*. Physica D. 239, 1387–1401 (2010).

[25] T. Hmidi and M. Zerguine : *Vortex patch for stratified Euler equations*. Commun. Math. Sci.Vol. 12, no. 8, 1541–1563 (2014).

[26] T. Hmidi and S. Keraani : *Inviscid limit for the two-dimensional Navier-Stokes equation in a critical Besov space*.Asymptotic Analysis, 53 (3), 125–138 (2007)

[27] T. Hmidi, S. Keraani and F. Rousset : *Global well-posedness for a Navier-Stokes-Boussinesq system with critical dissipation*. J. Differential Equations 249, 2147–2174 (2010).

[28] T. Hmidi and F. Rousset : *Global well-posedness for the Euler-Boussinesq system with axisymmetric data*. J. Funct. Anal. 260 (3), 745–796 (2011).

[29] A. Larios, E. Lunasin and E. S. Titi : *Global well-posedness for the $2d-$Boussinesq system without heat diffusion and with either anisotropic viscosity or inviscid voigt$-\alpha$ regularization*. Journal of Differential Equations, 255, 2636-2654 (2013).

[30] P.-G. Lemarié-Rieusset : *Recent Developments in the Navier Stokes Problem*. Chapman & Hall/CRC Research Notes in Mathematics, 431, Chapman & Hall/CRC, Boca Raton, FL, 2002.

[31] D. Li and X. Xu : *Global well-posedness of an inviscid* 2*D Boussinesq system with nonlinear thermal diffusivity*. Dyn. Partial Differ. Equ. 10 (3), 255–265 (2013).

[32] H. Li, R. Pan and W. Zhang : *Initial boundary value problem for* 2*D Boussinesq equations with temperature-dependent diffusion*. J. Hyperbolic Differ. Equ. 12(3), 469–488 (2015).

[33] S. A. Lorca and J. L. Boldrini : *The initial value problem for a generalized Boussinesq model : regularity and global existence of strong solutions*. Mat. Contemp. 110, 71–94 (1996).

[34] S. A. Lorca and J. L. Boldrini : *The initial value problem for a generalized Boussinesq model*. Nonlinear Anal., 36 (4), 457–480 (1999).

[35] H. Meddour : *Local stability of geometric structures for Boussinesq system with zero viscosity*. Accepted in

[36] H. Meddour and M. Zerguine : *Optimal rate of convergence in stratified Boussinesq system*. Dynamics of PDE, Vol. 15, no. 4, 235–263 (2018).

[37] M. Paicu and N. Zhu : *On the Yudovitch's type solutions for* 2*D Boussinesq system with thermal diffusivity.* Discrete and Continuous Dynamical Systems. In press. ffhal-02500672f.

[38] J. Pedlosky : *Ocean circulation theory*. Springer-Verlag Berlin Heidelberg 1996.

[39] C. Miao and L. Xue : *On the global well-posedness of a class of Boussinesq- Navier-Stokes systems*. arXiv :0910.0311v1.

[40] P. Serfati : *Une preuve directe d'existence globale des vortex patches 2D*. C. R. Acad. Sci. Paris Sér. I Math, 318, No. 6, 515-518 (1994).

[41] Y. Sun and Z. Zhang : *Global regularity for the initial-boundary value problem of the* 2*D Boussinesq system with variable viscosity and thermal diffusivity*. J. Differential Equations, 255 (6) : 1069-1085 (2013).

[42] T. Tao and L. Zhang : Hölder continuous solutions of Boussinesq equation with compact support. J. Funct. Anal., 272 (10), 4334–4402 (2017).

[43] H. von Tippelkirch : *Uber Konvektionszeller insbesondere in flussigen Schefel*. Beitrage Phys. Atmos., 20, 37-54 (1956).

[44] S. Wang and Z. Zhang : *Global well-posedness for the* 2*D Boussinesq system with the temperature-dependent viscosity and thermal diffusivity*. Adv. Math. 228, 43–62 (2011).

[45] J. Wu and X. Xu. *Well-posedness and inviscid limits of the Boussinesq equations with fractional Laplacian dissipation*. Nonlinearity, 27 (9), 2215–2232 (2014).

[46] J. Wu, X. Xu, L. Xue and Z. Ye. *Regularity results for the* 2*D Boussinesq equations with critical or supercritical dissipation*. Commun. Math. Sci. 14 (7), 1963–1997 (2016).

[47] E. Weinan and C. Shu : Small-scale structures in Boussinesq convection. Phys. Fluids 6, 49-58 (1994).

[48] V. I. Yudovich : *Non-stationnary flows of an ideal incompressible fluid*. Zhurnal Vych Matematika, 3, 1032-106 (1963).

[49] M. Zerguine : *The regular vortex patch for stratified Euler equations with critical fractional dissipation*. J. Evol. Equ. 15, 667-698 (2015).