VANISHING VISCOSITY FOR THE NAVIER-STOKES BOUSSINESQ SYSTEM

Youssouf Maafa, Mohamed Zerguine
University of Batna -2-, LEDPA

ABSTRACT
This paper deals with the local well-posedness in time for the Navier-Stokes Boussinesq equations in two dimensions in the framework of a smooth vortex patch. Furthermore, we provide the inviscid limit for the velocity and the density.

1. INTRODUCTION

The system of the Navier-Stokes Boussinesq with viscosity \( \mu > 0 \) given by the coupled equation,

\[
\begin{aligned}
\partial_t v_\mu + v_\mu \cdot \nabla v_\mu - \mu \Delta v_\mu + \nabla p &= \theta_\mu \vec{e}_2 \quad \text{if} \ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\
\partial_t \theta_\mu + v_\mu \cdot \nabla \theta_\mu &= 0 \quad \text{if} \ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\
\text{div} v_\mu &= 0, \\
(v_\mu, \theta_\mu)|_{t=0} &= (v_0, \theta_0),
\end{aligned}
\]

(NSB\(_\mu\))

where \( v = (v^1, v^2) \mathbb{R}^2 \) refers to the velocity vector field located in position \( x \in \mathbb{R}^2 \) at a time \( t \) which assumed to be incompressible, the scalar function \( \theta(t, x) \in \mathbb{R}_+ \) denotes the temperature or the density, \( p(t, x) \in \mathbb{R} \) is the pressure which relates \( v \) and \( \theta \) through an elliptic equation. \( \theta_\mu \vec{e}_2 \) is the buoyancy force in the direction \( \vec{e}_2 = (0, 1) \).

Taking the curl operator to the momentum equation in (NSB\(_\mu\)) we get

\[
\begin{aligned}
\partial_t \omega_\mu + v_\mu \cdot \nabla \omega_\mu - \mu \Delta \omega_\mu &= \partial_1 \theta_\mu, \\
\partial_t \theta_\mu + v_\mu \cdot \nabla \theta_\mu &= 0, \\
(\theta_\mu, \omega_\mu)|_{t=0} &= (\theta_0, \omega_0).
\end{aligned}
\]

(VD\(_\mu\))

Our goal is to prove that the system (NSB\(_\mu\)) is locally well-posed whenever the initial vorticity is a smooth vortex patch, that is \( \omega_0 = \chi_{\Omega_0} \), with the boundary \( \partial \Omega_0 \) is a Jordan curve with \( C^{2+\varepsilon} \) regularity, \( 0 < \varepsilon < 1 \). In addition, we prove the local persistence of geometric structures as follows, equivalently the image \( \Omega_t = \Psi_\mu(t, \Omega_0) \) keeps its initial regularity, with \( \Psi_\mu \) is the flow generated by the velocity \( v_\mu \).

\[
\begin{aligned}
\partial_t \Psi_\mu(t, x) &= v(t, \Psi_\mu(t, x)), \\
\Psi_\mu(0, x) &= x.
\end{aligned}
\]

Our second task is to study the inviscid limit of the system (NSB\(_\mu\)) towards the system (EB) given by the so-called Euler Boussinesq

\[
\begin{aligned}
\partial_t v + v \cdot \nabla v + \nabla p &= \theta \vec{e}_2 \quad \text{if} \ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\
\partial_t \theta + v \cdot \nabla \theta &= 0 \quad \text{if} \ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^2, \\
\text{div} v &= 0, \\
(v, \theta)|_{t=0} &= (v^0, \theta^0).
\end{aligned}
\]

(EB)
and we evaluate the rate of convergence between velocities, densities and vortices when the viscosity.

Let us briefly mention some results related to the classical Euler equation. J.Y. Chemin [11] showed that if the initial boundary \( \partial \Omega_0 \) is \( C^{1+\varepsilon} \), for \( \varepsilon \in (0,1) \) then \( \partial \Omega_0 \) is still of class \( C^{1+\varepsilon} \). In this context, Hmidi and Zerguine [25] extended the result of [11] to the stratified Euler equation. For other connected subjects in different situations we refer the reader to \([1,12,14,13,16,19,20,35]\) and the references therein. On the other hand Meddour and Zerguine [35] explored the inviscid limit of the Navier-stokes Boussinesq system to the Stratified Euler equation in the vortex patch setting. Inspired by the works \([11,25,35]\), we are mainly interested by studying the Navier-Stokes Boussinesq equations. The first result deals with the local existence and the local persistence of geometric structures of the system \([NSB]\). In particular, we have the following Theorem.

**Theorem 1.1** Let \( 0 < \varepsilon < 1, \eta \in (1,\infty) \) and \( X_0 \) be a family of admissible vector fields and \( v_0^\mu \) be a free-divergence vector field in the sense that \( \omega^0 \in L^\varepsilon \cap C^{\varepsilon}(X_0) \). Let \( \theta^0_\mu \in L^2 \cap C^{\varepsilon+1}(X_0) \) with \( \nabla \theta^0_\mu \in L^a \), then for \( \mu \in [0,1] \) the system \([NSB_\mu]\) admits a unique global solution

\[
(v_\mu, \theta_\mu) \in L^\varepsilon([0,T];\text{Lip}) \times L^\varepsilon([0,T];\text{Lip} \cap L^2).
\]

More precisely,

\[
\|v_\mu\|_{L^\varepsilon L^\varepsilon} \leq C_0 e^{e^\mu}.
\]

Furthermore,

\[
\|\omega_\mu\|_{L^\varepsilon C(X)} + \|X_\mu\|_{L^\varepsilon C(X)} + \|\Psi_\mu\|_{L^\varepsilon C^{-1}(X)} \leq C_0 e^{e^\mu}.
\]

The second result of this paper deals with the inviscid limit for the system \([NSB_\mu]\) to the stratified-Euler system. More precisely, we have

**Theorem 1.2** Let \((v_\mu, \theta_\mu), (v, \theta)\), be the solution of the \([NSB_\mu]\), \([EB]\) respectively with the same initial data satisfies the condition of Theorem 1.1 such that \( \theta^0_\mu = \theta^0 = 1_{\Omega_0} \) where \( \Omega_0 \) is simply connected bounded domain. Then for all \( t \geq 0, \mu \in [0,1] \), we have

\[
\|v_\mu(t) - v(t)\|_{L^2} + \|\theta_\mu(t) - \theta(t)\|_{L^2} \leq C_0(\mu t)^{1/2}.
\]

2. TOOL BOX

### 2.1. Function spaces

Let \( \chi, \varphi \in \mathcal{S}(\mathbb{R}^2) \times \mathcal{S}(\mathbb{R}^2) \) be a radial cut-off functions be such that \( \text{supp} \, \chi \subset \{ \xi \in \mathbb{R}^2 : \| \xi \| \leq 1 \} \) and \( \text{supp} \, \varphi(\xi) \subset \{ \xi \in \mathbb{R}^2 : 1/2 \leq \| \xi \| \leq 2 \} \), so that

\[
\chi(\xi) + \sum_{q \geq 0} \varphi(2^{-q} \xi) = 1.
\]

Through \( \chi \) and \( \varphi \), the Littlewood-Paley or frequency cut-off operators \((\Delta_q)_{q \geq 1}\) and \((\tilde{\Delta}_q)_{q \geq 1}\) are defined for \( u \in \mathcal{S}'(\mathbb{R}^2) \)

\[\Delta_{-1}u = \chi(D)u, \quad \Delta_q u = \varphi(2^{-q}D)u \quad \text{for} \quad q \in \mathbb{N}, \quad \tilde{\Delta}_q u = \varphi(2^{-q}D)u \quad \text{for} \quad q \in \mathbb{Z}.
\]

where in general case \( f(D) \) stands the pseudo-differential operator \( u \mapsto \mathcal{F}^{-1}(f(D)u) \) with constant symbol. The lower frequencies sequence \((S_q)_{q \geq 0}\) is defined for \( q \geq 0 \),

\[
S_q u \triangleq \sum_{j \geq q-1} \Delta_j u.
\]

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In accordance of the previous properties we derive the well-known decomposition of unity

\[ u = \sum_{q \in \mathbb{Z}} \Delta_q u, \quad u = \sum_{q \in \mathbb{Z}} \Delta_q u. \]

The results currently available allow us to define the inhomogeneous Besov denoted \( B^p_{p,r} \) (resp. \( B^p_{p,r} \)) and defined in the following way.

**Definition 2.1** For \((p, r, s) \in [1, +\infty)^2 \times \mathbb{R}\), the inhomogeneous Besov spaces \( B^s_{p,r} \) (resp. homogeneous Besov spaces \( B^s_{p,r} \)) are defined by

\[
\begin{align*}
B^s_{p,r} &= \{ u \in \mathcal{S}'(\mathbb{R}^2) : \| u \|_{B^s_{p,r}} < +\infty \}, \\
B^s_{p,r} &= \{ u \in \mathcal{S}'(\mathbb{R}^2) : \| u \|_{B^s_{p,r}} < +\infty \},
\end{align*}
\]

where \( \mathcal{P} \) refers to the set of polynomial functions in \( \mathbb{R}^2 \) so that

\[
\| u \|_{B^s_{p,r}} \triangleq \begin{cases} \\
\left( \sum_{q \ge -1} 2^{qs} \| \Delta_q u \|_{L^p}^r \right)^{1/r} & \text{if } r \in [1, +\infty], \\
\sup_{q \ge -1} 2^{qs} \| \Delta_q u \|_{L^p} & \text{if } r = +\infty.
\end{cases}
\]

and

\[
\| u \|_{B^s_{p,r}} \triangleq \begin{cases} \\
\left( \sum_{q \in \mathbb{Z}} 2^{qs} \| \Delta_q u \|_{L^p}^r \right)^{1/r} & \text{if } r \in [1, +\infty], \\
\sup_{q \in \mathbb{Z}} 2^{qs} \| \Delta_q u \|_{L^p} & \text{if } r = +\infty.
\end{cases}
\]

The Bernstein’s inequalities are listed in the following lemma.

**Lemma 2.2** There exists a constant \( C > 0 \) such that for \( 1 \le a \le b \le \infty \), for every function \( u \) and every \( q \in \mathbb{N} \cup \{ -1 \} \), we have

(i) \( \sup_{|\alpha|=k} \| \partial^\alpha S_q u \|_{L^b} \le C^k 2^{q(k+2(1/a-1/b))} \| S_q u \|_{L^a} \).

(ii) \( C^{-1} 2^q \| \Delta_q u \|_{L^b} \le \sup_{|\alpha|=k} \| \partial^\alpha \Delta_q u \|_{L^a} \le C^k 2^q \| \Delta_q u \|_{L^a} \).

As a consequence of Bernstein inequality, we have

**Proposition 2.3** For \((s, \tilde{s}, p_1, p_2, r_1, r_2) \in [1, \infty] \times \mathbb{R} \) with \( s \le \tilde{s}, p_1 \le p_2 \) and \( r_1 \le r_2 \), then we have

(i) \( B^s_{p,r} \hookrightarrow \tilde{B}^s_{p,r} \).

(ii) \( B^s_{p_1, r_1} (\mathbb{R}^2) \hookrightarrow \tilde{B}^s_{p_2, r_2} (\mathbb{R}^2) \).

Now, we state Bony’s decomposition [3] which allows us to split formally the product of two tempered distributions \( u \) and \( v \) into three pieces. More precisely, we have.

**Definition 2.4** For a given \( u, v \in \mathcal{S}' \) we have

\[ uv = T_{u,v} + T_u u + \mathcal{R}(u,v), \]

with

\[ T_{u,v} = \sum_q S_{q-1} u \Delta_q v, \quad \mathcal{R}(u,v) = \sum_q \Delta_q u \Delta_q v \quad \text{and} \quad \tilde{\Delta}_q = \Delta_{q-1} + \Delta_q + \Delta_{q+1}. \]

The mixed space-time spaces are stated as follows.

**Definition 2.5** Let \( T > 0 \) and \((s, \beta, p, r) \in [1, \infty]^3 \). We define the spaces \( L^p_T B^s_{p,r} \) and \( \tilde{L}^p_T B^s_{p,r} \) respectively by :

\[
\begin{align*}
L^p_T B^s_{p,r} &\triangleq \left\{ u : [0, T] \to \mathcal{S}' : \| u \|_{L^p_T B^s_{p,r}} = \left\| (2^{qs} \| \Delta_q u \|_{L^p})_t \right\|_{L^p_T} < \infty \right\}, \\
\tilde{L}^p_T B^s_{p,r} &\triangleq \left\{ u : [0, T] \to \mathcal{S}' : \| u \|_{\tilde{L}^p_T B^s_{p,r}} = \left\| (2^{qs} \| \Delta_q u \|_{L^p})_t \right\|_{L^p_T} < \infty \right\}.
\end{align*}
\]
The following result is a useful result in our approach. For the proof see [19] Corollary 1.

**Corollary 2.6** Given \( \varepsilon \in ]0,1[ \) and \( X \) be a vector field be such that \( X, \text{div}X \in \mathcal{C}^{\varepsilon} \). Then for \( f \) be a Lipschitz scalar function \( k \in \{1,2\} \) the following statement holds.

\[
\| (\partial_{k}X) \cdot \nabla f \|_{\mathcal{C}^{\varepsilon-1}} \leq C\| \nabla f \|_{L^{k}} \left( \| \text{div}X \|_{\mathcal{C}^{\varepsilon}} + \| X \|_{\mathcal{C}^{\varepsilon}} \right).
\]

### 2.2. Particle results

In this subsection, we give some preparatory results freely used throughout our analysis.

\[
\begin{aligned}
\begin{cases}
\partial a + v \cdot \nabla a - \mu \Delta a = g, \\
\lambda = 0 = a^{0}.
\end{cases}
\end{aligned}
\]

We start with the persistence of Besov regularity for (1) whose proof may be found in [4].

**Proposition 2.7** Let \( (s,r,p) \in ]1,1[\times]1,\infty[ \) and \( v \) be a smooth vector field in free-divergence. Assume that \( (a^{0},g) \in \text{B}_{p,r}^{s} \times \text{L}_{p,r}^{s} \). Then for every smooth solution \( a \) of (1) and \( t \geq 0 \) we have

\[
\| a(t) \|_{\text{B}_{p,r}^{s}} \leq C e^{\text{CV}(t)} \left( \| a^{0} \|_{\text{B}_{p,r}^{s}} + \int_{0}^{t} e^{-\text{CV}(\tau)} \| g(\tau) \|_{\text{B}_{p,r}^{s}} d\tau \right),
\]

with the notation

\[
V(t) = \int_{0}^{t} \| \nabla v(\tau) \|_{L^{r}} d\tau,
\]

where \( C = C(s) \) being a positive constant.

The statement of maximal regularity for (1) in mixed space-time Besov space is given by the following result. For the proof see [5].

**Proposition 2.8** Let \( (s,p_{1},p_{2},r) \in ]1,1[\times]1,\infty[ \times]1,\infty[ \) and \( v \) be a free-divergence vector field belongs to \( \text{L}_{p,r}^{s} \). Then there exists a constant \( C \geq 0 \), so that for every smooth solution \( a \) of (1) we have for all \( t \geq 0 \)

\[
\mu \frac{t}{2} \| a \|_{\text{B}_{p_{1},p_{2}}^{s}} \leq C e^{-\text{CV}(t)} \left( 1 + \mu t \right) \left( \| a^{0} \|_{\text{B}_{p_{1},p_{2}}^{s}} + \| g \|_{\text{L}_{p,r}^{s}} \right).
\]

Next, we have the classical Càdlàg Zygmund inequality.

**Proposition 2.9** Let \( p \in ]1,\infty[ \) and \( v \) be a free-divergence vector field whose vorticity \( \omega \in \text{L}^{p} \). Then \( \nabla v \in \text{L}^{p} \) and

\[
\| \nabla v \|_{\text{L}^{p}} \leq c \frac{p}{p-1} \| \omega \|_{\text{L}^{p}}.
\]

with \( C \) being a universal constant.

At this stage, we define the anisotropic Hölder spaces as follows

**Definition 2.10** Let \( \varepsilon \in ]0,1[ \). A family of vector fields \( X = (X_{\lambda})_{\lambda \in A} \) is said to be admissible if and only if the following assertions are hold.

1. **Regularity**: \( \forall \lambda \in A \quad X_{\lambda}, \text{div}X_{\lambda} \in \mathcal{C}^{\varepsilon} \).
2. **Non-degeneracy**: \( I(X) \triangleq \inf_{x \in \mathbb{R}^{a}} \sup_{\lambda \in A} | X_{\lambda}(x) | > 0. \)
Setting

\[ \|X_\lambda\|_{C^\varepsilon} \triangleq \|X_\lambda\|_{C^\varepsilon} + \|\text{div}X_\lambda\|_{C^\varepsilon}. \]

**Definition 2.11** Let \( X = (X_\lambda)_{\lambda \in \Lambda} \) be an admissible family. The action of each factor \( X_\lambda \) on \( u \in L^\infty \) is defined as the directional derivative of \( u \) along \( X_\lambda \) by the formula,

\[ \partial_{X_\lambda} u = \text{div}(uX_\lambda) - u\text{div}X_\lambda. \]

The concept of anisotropic Hölder space, will be noted by \( C^\varepsilon(X) \) is defined below.

**Definition 2.12** Let \( \varepsilon \in [0, 1] \) and \( X \) be an admissible family of vector fields. We say that \( u \in C^\varepsilon(X) \) if and only if:

(i) \( u \in L^\infty \) and satisfies

\[ \forall \lambda \in \Lambda, \partial_{X_\lambda} u \in C^{\varepsilon-1}, \sup_{\lambda \in \Lambda} \|\partial_{X_\lambda} u\|_{C^{\varepsilon-1}} < +\infty. \]

(ii) \( C^\varepsilon(X) \) is a normed space with

\[ \|u\|_{C^\varepsilon(X)} \triangleq \frac{1}{|\Pi|} \left( \|u\|_{L^\infty} \sup_{\lambda \in \Lambda} \|X_\lambda\|_{C^\varepsilon} + \sup_{\lambda \in \Lambda} \|\partial_{X_\lambda} u\|_{C^{\varepsilon-1}} \right). \]

The next result play a major role in the proof of our main results. We refer the reader to \[11\].

**Theorem 2.13** Let \( \varepsilon \in [0, 1] \) and \( X = (X_\lambda)_{\lambda \in \Lambda} \) be a family of vector fields as in Definition 2.12. Let \( v \) be a free-divergence vector field such that its vorticity \( \omega \) belongs to \( L^2 \cap C^\varepsilon(X) \). Then there exists a constant \( C \) depending only on \( \varepsilon \), such that

\[ \|\nabla v\|_{L^\infty} \leq C \left( \|\omega\|_{L^2} + \|\omega\|_{L^\infty} \log \left( e + \frac{\|\omega\|_{C^\varepsilon(X)}}{\|\omega\|_{L^\infty}} \right) \right). \]

According Danchin’s result \[12\], the class \( C^\varepsilon \) doesn’t covers only the vortex patch of the type \( \omega_0 = 1_{\Omega_0} \), but also encompass the so-called general vortex. Specifically, we have.

**Proposition 2.14** Let \( \Omega_0 \) be a \( C^{1+\varepsilon} \)-bounded domain, with \( 0 < \varepsilon < 1 \). Then for every function \( f \in C^\varepsilon \), we have

\[ f1_{\Omega_0} \in C^\varepsilon. \]

### 2.3. A priori estimates

In this part we shall give some a priori estimates for the velocity and the vorticity.

**Proposition 2.15** Let \( v_\mu \) be a smooth divergence-free vector field and \( \theta_\mu \) be a smooth solution of the equation (\texttt{?}). Then the following assertions are hold.

(i) For \( p \in [1, \infty] \) and \( t \geq 0 \) we have

\[ \|\theta_\mu(t)\|_{L^p} \leq \|\theta_\mu^0\|_{L^p}. \]

(ii) For \( p \in [1, \infty] \) and \( t \geq 0 \) we have

\[ \|\nabla \theta_\mu(t)\|_{L^p} \leq \|\nabla \theta_\mu^0\|_{L^p} e^{C_{V_\mu}(t)}. \]

with \( V_\mu(t) = \int_0^t \|\nabla v_\mu(\tau)\|_{L^\infty} d\tau. \)
(iii) For $p \in [1, \infty]$ and $t \geq 0$ we have
\[ \| \omega_\mu(t) \|_{L^p} \leq C_0 e^{C V_\mu(t)}. \]
with $V(t) = \int_0^t \| \nabla \omega_\mu(\tau) \|_{L^p} d\tau$.

**Proof.** (i) According to [1] we can express the density $\theta_\mu(t)$ by the initial value $\theta_\mu^0$ and the flow $\Psi$ as follows
\[ \theta_\mu(t, x) = \theta_\mu^0(\Psi^{-1}(t, x)). \]
Taking the $L^p$-norm to this equation and thanks to incompressible condition we infer that
\[ \| \theta_\mu(t) \|_{L^p} \leq \| \theta_\mu^0 \|_{L^p}. \]

(ii) Taking the partial derivative $\partial_j$ to $\theta_\mu$ - equation to obtain
\[ \partial_j \partial_j \theta + \nu \nabla \theta = -\partial_j \theta \cdot \nabla \nu, \]
The $L^p$-estimate for the above gives
\[ \| \nabla \theta(t) \|_{L^p} \leq \| \nabla \theta_0 \|_{L^p} + \int_0^t \| \nabla \theta(\tau) \|_{L^p} \| \nabla \nu(\tau) \|_{L^p} d\tau. \]
The Gronwall’s inequality implies that
\[ \| \nabla \theta(t) \|_{L^p} \leq \| \nabla \theta_0 \|_{L^p} e^{C V(t)}. \]

(iii) The $L^p$-estimate for the $\omega_\mu$ equation gives
\[ \| \omega_\mu(t) \|_{L^p} \leq \| \omega_\mu^0 \|_{L^p} + \int_0^t \| \nabla \theta(\tau) \|_{L^p} \| \nabla \nu(\tau) \|_{L^p} d\tau. \]
Combining the last two estimates we find the desired result. ■

**Proof of Theorem [14]** The existence part of the theorem is classical and can be done for example by using a standard recursive method, see, e.g. [19]. We will control the quantities $\| \nabla \nu(t) \|_{L^p}$ and $\| \omega_\mu(t) \|_{C^0(X)}$ for every $t \geq 0$. For this aim, appalling the operator $\partial X_{\mu} \lambda$ to $\omega_\mu$ equation, we have
\[ (\partial_t + \nu \cdot \nabla - \mu \Delta) \partial X_{\mu} \lambda \omega_\mu = X_{\mu} \lambda \cdot \nabla \partial_\mu \theta_\mu - \mu [\Delta, X_{\mu} \lambda] \omega_\mu. \]
For commutator $\mu [\Delta, X_{\mu} \lambda] \omega_\mu$. From Bony’s decomposition, we write
\[ \mu [\Delta, X_{\mu} \lambda] \omega_\mu = \mathfrak{A} + \mu \mathfrak{B}, \]
with
\[ \mathfrak{A} \triangleq 2 \mu T_{X_{\mu} \lambda} \partial_\mu \omega_\mu + 2 \mu T_\partial \nabla \omega_\mu \nabla X_{\mu} \lambda + \mu T_{X_{\mu} \lambda} \partial_\mu \omega_\mu + \mu T_\partial \omega_\mu \Delta X_{\mu} \lambda, \]
and
\[ \mathfrak{B} \triangleq 2 \mathcal{R}(\nabla X_{\mu} \lambda, \partial_\mu \omega_\mu) + \mathcal{R}(\Delta X_{\mu} \lambda, \partial_\mu \omega_\mu), \]
According to Theorem 3.38 page 162 in [13], we have
\[ \| \partial X_{\mu} \omega_\mu \|_{L^\infty} \leq C e^{C V_\mu(t)} \left( \| \partial X_{\mu} \omega_\mu^0 \|_{C^0} + (1 + \mu t) \| \mathfrak{A} \|_{L^\infty} + \mu \| \mathfrak{B} \|_{L^\infty} + \| \partial X_{\mu} \partial_\mu \theta_\mu \|_{L^\infty} \right). \]
From [20] we have the following estimate
\[ \| \mathfrak{A} \|_{L^\infty} \leq C \| \omega \|_{L^\infty} \| X_{\mu} \|_{L^\infty}. \]

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Thanks to Proposition 2.15, we get
\[ \| \mathcal{A} \|_{L^\infty \Theta^-} \leq C_0 e^{CV_\delta(t)} \| X_\lambda \|_{L^\infty \Theta^s}. \]  
(5)

Again from 2.8 we have the following estimate
\[ \| \mathcal{B} \|_{L^\infty \Theta^-} \leq C \| \omega \|_{L^1 B_{\infty,m}^0} \| X_\lambda \|_{L^\infty \Theta^s}. \]  
(6)

For the term \( \| \omega_\mu \|_{L^1 B_{\infty,m}^0} \), we use the Proposition 2.8 for \( a = \omega_\mu, \delta_1 \theta_\mu, r = 1, s = 0 \), and \( p_1 = p_2 = \infty \), we obtain
\[ \mu \| \omega_\mu \|_{L^1 B_{\infty,m}^0} \leq C e^{CV_\delta(t)} (1 + \mu t) \left( \| \omega_\mu^0 \|_{B_{\infty,m}^0} + \int_0^t \| \delta_1 \theta_\mu (\tau) \|_{B_{\infty,m}^0} d \tau \right). \]

The embedding \( L^m \hookrightarrow B_{\infty,m}^0 \) implies that
\[ \mu \| \omega_\mu \|_{L^1 B_{\infty,m}^0} \leq C e^{CV_\delta(t)} (1 + \mu t) \left( \| \omega_\mu^0 \|_{L^\infty} + \| \nabla \theta_\mu \|_{L^1} \right). \]

Using Proposition 2.8 we obtain
\[ \mu \| \omega_\mu \|_{L^1 B_{\infty,m}^0} \leq C_0 e^{CV_\delta(t)} (1 + \mu t)(1 + t) \]

Plugging the last estimate into (7), we infer that
\[ \| \mathcal{B} \|_{L^\infty \Theta^-} \leq C_0 e^{CV_\delta(t)} (1 + \mu t)(1 + t) \| X_\lambda \|_{L^\infty \Theta^s}. \]  
(7)

To treat the quantity \( \| \partial_{X_\lambda} \delta_1 \theta_\mu \|_{L^1 \Theta^s} \) we note that
\[ \partial_{X_\lambda} \delta_1 \theta_\mu = \delta_1 (\partial_{X_\lambda} \theta_\mu) - \partial_{\delta_1 X_\lambda} \theta_\mu. \]  
(8)

It follows, from taking the \( C^s \) norm to this equation
\[ \| \partial_{X_\lambda} \delta_1 \theta_\mu (\tau) \|_{C^s} \leq \| \partial_1 (\partial_{X_\lambda} \theta_\mu) (\tau) \|_{C^s} + \| (\partial_1 X_\lambda, \cdot \nabla \theta_\mu (\tau) \|_{C^s}. \]

Moreover using the fact \( \partial_1 : C^s \rightarrow C^s \) is a continuous map and Corollary 2.6 we get
\[ \| \partial_{X_\lambda} \delta_1 \theta_\mu (\tau) \|_{C^s} \leq \| \partial_{X_\lambda} \theta_\mu (\tau) \|_{C^s} + \| \nabla \theta_\mu (\tau) \|_{L^\infty} \| X_\lambda, \cdot C^s e^{CV_\delta(t)}. \]

From Proposition 2.15 and Proposition 2.15 we find
\[ \| \partial_{X_\lambda} \delta_1 \theta_\mu (\tau) \|_{L^1 \Theta^s} \leq \| \partial_{X_\lambda} \theta_\mu^0 \|_{C^s e^{CV_\delta(t)}} + C_0 \| X_\lambda, \cdot C^s e^{CV_\delta(t)} \]  
(9)

Summing (5),(7),(9) and punting them in (4), such that \( \mu \in [0,1] \) we infer that
\[ \| \partial_{X_\lambda} \omega_\mu \|_{L^1 \Theta^-} \leq C_0 e^{CV_\delta(t)} (1 + t^2) \| X_\lambda, \cdot C^s \]  
(10)

On other hand bound by using Proposition 2.7 we find
\[ \| X_\lambda, \cdot C^s \leq C e^{CV_\delta(t)} \left( \| X_\lambda, \cdot C^s + \int_0^t e^{-CV_\delta(t)} \| \partial_{X_\lambda} \nu_\mu (\tau) \|_{C^s} d \tau \right). \]  
(11)

We use the following result which its proof can be found in 5,11
\[ \| \partial_{X_\lambda} \nu_\mu (t) \|_{C^s} \leq C (\| \nabla \nu_\mu (t) \|_{L^\infty} \| X_\lambda, \cdot C^s + \| \omega_\mu (t) \|_{C^s}). \]
Thanks to (10), we find
\[ \| \partial_{x_i} v_\mu(t) \|_{L^\infty} \leq C \sup_{\lambda \in \Lambda} \| X_{0,\lambda} \|_{L^\infty} \left( \| \nabla v_\mu(t) \|_{L^2} + C_0 e^{C V_\mu(t)(t)} (1 + t)^2 \right). \] (12)

Plugging (12) in (11), we get
\[ \| X_{t,\lambda} \|_{L^\infty} \leq C e^{C V_\mu(t)} \left( \| X_{0,\lambda} \|_{L^\infty} + C_0 \int_0^t e^{-C V_\mu(\tau)} \| X_{t,\lambda} \|_{L^\infty} \left( \| \nabla v_\mu(\tau) \|_{L^2} + (1 + \tau)^2 \right) d\tau \right). \]

For the term \( \| \text{div} X_{t,\lambda} \|_{L^\infty} \) by applying Proposition 2.7 to \( (\partial_v + v \cdot \nabla) \text{div} X_{t,\lambda} = 0 \), we get
\[ \| \text{div} X_{t,\lambda} \|_{L^\infty} \leq C e^{C V_\mu(t)} \| \text{div} X_{0,\lambda} \|_{L^\infty}. \] (13)

Combining the last two estimates we have
\[ e^{-C V_\mu(t)} \| X_{t,\lambda} \|_{L^\infty} \leq C \left( \| X_{0,\lambda} \|_{L^\infty} + C_0 \int_0^t e^{-C V_\mu(\tau)} \| X_{t,\lambda} \|_{L^\infty} \left( \| \nabla v_\mu(\tau) \|_{L^2} + (1 + \tau)^2 \right) d\tau \right). \]

The Gronwall’s inequality gives
\[ \| X_{t,\lambda} \|_{L^\infty} \leq C_0 e^{C V_\mu(t)} e^{C t^3}. \] (15)

Gathering (10) and (15), one has
\[ \| \partial_{x_i} \omega_\mu(t) \|_{L^\infty} \leq C_0 e^{C V_\mu(t)} e^{C t^3}. \]

Moreover, from the last two estimates and Proposition 2.15 we get
\[ \| \partial_{x_i} \omega_\mu(t) \|_{L^\infty} + \| \omega_\mu(t) \|_{L^2} \| X_{t,\lambda} \|_{L^\infty} \leq C_0 e^{C V_\mu(t)} e^{C t^3}. \] (16)

Now, we recall that
\[ \| \omega \|_{L^\infty} \triangleq \frac{1}{I(X_t)} \left( \| \omega \|_{L^\infty} \sup_{\lambda \in \Lambda} \| X_{t,\lambda} \|_{L^\infty} \right) \sup_{\lambda \in \Lambda} \| \partial_{x_i} \omega \|_{L^\infty}. \] (17)

To control the term \( I(X_t) \) we apply the derivative in time to the quantity \( \partial_{x_0} \Psi \), it follows
\[ \left\{ \begin{array}{rcl}
\partial_t \partial_{x_0} \Psi(t,x) &=& \nabla v(t,\Psi(t,x)) \partial_{x_0} \Psi(t,x) \\
\partial_{x_0} \Psi(0,x) &=& X_{0,\lambda}. \end{array} \right. \]

The time reversibility of the previous equation and Gronwall’s inequality ensure that
\[ |X_{0,\lambda}(x)| \leq |\partial_{x_0} \Psi(t,x)| e^{V_\mu(t)}. \]

From (ii) in Definition 2.10 we get
\[ I(X_t) \geq I(X_0) e^{-V_\mu(t)} > 0. \] (18)

Thanks to (16), (17) and (18), we have
\[ \| \omega_\mu(t) \|_{L^\infty} \leq C_0 e^{C t^3} e^{C V_\mu(t)}. \] (19)

According to Theorem 2.13 and Proposition 2.15, we obtain
\[ \| \nabla v_\mu(t) \|_{L^2} \leq C \left( C_0 + C_0 \log \left( e + \frac{\| \omega_\mu(t) \|_{L^\infty}^{\infty}}{\| \omega_\mu(t) \|_{L^2}} \right) \right). \]
The monotonicity of function $x \mapsto \log(e + \frac{x}{2})$ gives
\[
\|\nabla v_\mu(t)\|_{L^\infty} \leq C_0 \left( C_0 + C_0 \log \left( e + \frac{\|\nabla u(t)\|_{C^1(X)}}{\|\nabla u(t)\|_{L^\infty}} \right) \right).
\]

It follows from (19) that
\[
\|\nabla v_\mu(t)\|_{L^\infty} \leq C_0 \left( (1+t)^3 + \int_0^t \|\nabla v_\mu(\tau)\|_{L^\infty} d\tau \right).
\]

Again, Gronwall’s inequality gives
\[
\|\nabla v_\mu(t)\|_{L^\infty} \leq C_0 e^{C_0 t}.
\]

Together with (19), we have
\[
\|\nabla v_\mu(t)\|_{C^1(X)} \leq C_0 e^{C_0 t}.
\]

To control the term $\Psi_\mu$ in $C^0(X)$, we recall that $\partial T \Psi_\mu(t) = X_{\xi, \lambda} \ast T \Psi_\mu(t)$. Thus, we get
\[
\|X_{\xi, \lambda} \ast T \Psi_\mu(t)\|_{C^0} \leq \|X_{\xi, \lambda}\|_{C^0} \|\nabla \Psi_\mu(t)\|_{L^\infty} \leq \|X_{\xi, \lambda}\|_{C^0} e^{C_0 t},
\]

where, we have used $\|\nabla \Psi_\mu(t)\|_{L^\infty} \leq e^{C_0 t}$. Consequently,
\[
\|\Psi_\mu(t)\|_{C^0(X)} \leq C_0 e^{C_0 t}.
\]

The proof of Theorem 1.1 is finished.

3. INVIScid LIMIT

Proof of Theorem 1.2

Taking the difference between (NSBE) and (EB), by setting $U = v_\mu - v, \Theta = \theta_\mu - \theta$ and $\Phi = p_\mu - p$ we find out that the triplet $(U, \Theta, \Phi)$ governs the following evolution system.

\[
\begin{cases}
\partial_t U + v_\mu \cdot \nabla U - \mu \Delta U = \Delta v - \nabla \Phi + \Theta H_2 - U \cdot \nabla v,
\partial_t \Theta + v_\mu \cdot \nabla \Theta = -U \cdot \nabla \theta,
\nabla \cdot U = 0,
(U, \Theta)_{t=0} = (U_0, \Theta_0).
\end{cases}
\]

Multiplying the first equation in the system (D) by $U$ and integrating by part over $\mathbb{R}^2$, such that $\text{div} v_\mu = \text{div} v = 0$ and Hölder’s inequality ensure that
\[
\frac{1}{2} \frac{d}{dt} \|U(t)\|_{L^2}^2 + \mu \|\nabla U(t)\|_{L^2}^2 \leq \mu \|\nabla v(t)\|_{L^2}^2 + \|\nabla v(t)\|_{L^\infty} \|U(t)\|_{L^2} + \|\Theta(t)\|_{L^2} \|U(t)\|_{L^2}.
\]

Young inequality implies that
\[
\frac{1}{2} \frac{d}{dt} \|U(t)\|_{L^2}^2 + \mu \|\nabla U(t)\|_{L^2}^2 \leq \frac{\mu}{2} \|\nabla v(t)\|_{L^2}^2 + \frac{\mu}{2} \|\nabla U(t)\|_{L^2}^2 + \left( \|U(t)\|_{L^2}^2 + \|\Theta(t)\|_{L^2}^2 \right) \left( \|\nabla v(t)\|_{L^\infty} + 1 \right).
\]

Integrating in time this inequality, we find
\[
\|U(t)\|_{L^2}^2 + \frac{\mu}{2} \|\nabla U(t)\|_{L^2}^2 \leq \|U(0)\|_{L^2}^2 + \frac{\mu}{2} \|\nabla v(0)\|_{L^2}^2 + \int_0^t \left( \|U(\tau)\|_{L^2}^2 + \|\Theta(\tau)\|_{L^2}^2 \right) \left( \|\nabla v(\tau)\|_{L^\infty} + 1 \right) d\tau.
\]

Similarly for $\Theta$–equation, we have
\[ \frac{1}{2} \frac{d}{dt} \| \Theta(t) \|^2_{L^2} \leq \| \nabla \theta(t) \|_{L^\infty} \| U(t) \|^2_{L^2}. \]

Integrating in time over \([0, t]\), it follows
\[ \| \Theta(t) \|^2_{L^2} \leq \| \Theta(0) \|^2_{L^2} + \int_0^t \| \nabla \theta(\tau) \|_{L^\infty} \| U(\tau) \|^2_{L^2} d\tau. \] (24)

Gathering (23) and (24), one has
\[ \Pi(t) \lesssim \Pi(0) + \int_0^t \Pi(\tau) \left( 1 + \| \nabla \theta(\tau) \|_{L^\infty} + \| \nabla v(\tau) \|_{L^\infty} \right) d\tau \]
\[ + \mu \| \nabla v \|_{L^2}^2, \] (25)

with \( \Pi(t) = \| v_\mu(t) - v(t) \|^2_{L^2} + \| \theta_\mu(t) - \theta(t) \|^2_{L^2}, \) so Gronwall’s inequality gives
\[ \Pi(t) \leq C e^{\frac{t}{\mu} + \| v_\mu(t) - v(t) \|^2_{L^2} + \| \theta_\mu(t) - \theta(t) \|^2_{L^2}} \left( \Pi(0) + \mu \| \nabla v_\mu \|_{L^2}^2 \right). \]

Again, Hölder’s inequality in time variable and using Proposition 2.9 we get
\[ \Pi(t) \leq C e^{\frac{t}{\mu} + \| v_\mu(t) - v(t) \|^2_{L^2} + \| \theta_\mu(t) - \theta(t) \|^2_{L^2}} \left( \Pi(0) + \mu \| \omega_\mu \|_{L^2} \right). \]

Thanks to Theorem and Proposition 2.15, we find that
\[ \Pi(t) \leq C_0 e^{C_0 (\mu t)}. \]

Consequently,
\[ \| v_\mu(t) - v(t) \|^2_{L^2} + \| \theta_\mu(t) - \theta(t) \|^2_{L^2} \leq C_0 (\mu t)^{\frac{1}{2}}. \]

This completes the proof of Theorem 1.2.

4. REFERENCES


[35] H. Meddour : Local stability of geometric structures for Boussinesq system with zero viscosity. Accepted in


