I-PACKING AND PACKING COLORING OF GENERALIZED PETERSON GRAPHS

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ABSTRACT

An i-packing in a graph G is a subset of vertices of $V(G)$. The cardinality of the largest i-packing is called the i-packing number and is noted $\rho_i(G)$
The packing chromatic number $\chi_\rho(G)$ of a graph G is the smallest integer k such that its set of vertices $V(G)$ can be partitioned into k disjoint subsets $V_1, \ldots, V_k$, in such a way that every subset V_i is an i packinh for every i, $1 \leq i \leq k$.

We determine the exact value or upper and lower bounds of the i-packing and the packing chromatic number number of Petersen generalized graphs.

Keywords : *i*-Packing ; packing coloring ; *i*-packing number ; Packing chromatic number ; Cubic graphs ; Generalized Petersen graph.

1. INTRODUCTION

All the graphs we consider are simple and loopless. For a graph G, we denote by $V(G)$ its set of vertices and by $E(G)$ its set of edges. The *distance* $d_G(u,v)$, or simply $d(u,v)$, when G is clear from the context, between vertices u and v in G is the length (number of edges) of a shortest path joining u and v. The *diameter* of G is the maximum distance between two vertices of G. We denote by P_n, $n \geq 1$, the path of order n and by C_n, $n \geq 3$, the cycle of order n.
An i-packing in a graph G is a subset of vertices of $V(G)$. The cardinality of the largest i-packing is called the i-packing number and is noted $\rho_i(G)$

A *packing k-coloring* of G is a mapping $\pi : V(G) \to \{1,\ldots,k\}$ such that, for every two distinct vertices u and v, $\pi(u) = \pi(v) = i$ implies $d(u,v) > i$. The *packing chromatic number* $\chi_\rho(G)$ of G is then the smallest k such that G admits a packing k-coloring. In other words, $\chi_\rho(G)$ is the smallest integer k such that $V(G)$ can be partitioned into k disjoint subsets $V_1, \ldots, V_k$, in such a way that every two vertices in V_i are at distance greater than i in G for every i, $1 \leq i \leq k$.
Packing coloring of graphs has been introduced by Goddard, Hedetniemi, Hedetniemi, Harris and Rall [**?**] under the name *broadcast coloring* and has been studied by several authors in recent years. Several papers deal with the packing chromatic number of certain classes of undirected graphs such as trees [**?**, **?**, **?**, **?**], lattices [**?**, **?**, **?**, **?**, **?**], Cartesian products [**?**, **?**, **?**], distance graphs [**?**, **?**, **?**] or hypercubes [**?**, **?**, **?**]. Complexity issues of the packing coloring problem were adressed in [**?**, **?**, **?**, **?**].

Let H be a subgraph of G. Since $d_G(u,v) \leq d_H(u,v)$ for any two vertices $u,v \in V(H)$, the restriction to $V(H)$ of any packing coloring of G is a packing coloring of H. This property obviously holds for digraphs as well. Hence, having packing chromatic number at most k is a hereditary property :

Proposition 1 (Goddard, Hedetniemi, Hedetniemi, Harris and Rall [?])
Let G and H be two graphs. If H is a subgraph of G, then $\chi_\rho(H) \leq \chi_\rho(G)$.

The packing chromatic number of paths and cycles has been determined by Goddard *et al.* :

Theorem 2 (Goddard, Hedetniemi, Hedetniemi, Harris and Rall [?])

1. *For every* $n \geq 1$, $\chi_\rho(P_n) \leq 3$. *Moreover,* $\chi_\rho(P_n) = 1$ *if and only if* $n = 1$ *and* $\chi_\rho(P_n) = 2$ *if and only if* $n \in \{2,3\}$.
2. *For every* $n \geq 3$, $3 \leq \chi_\rho(C_n) \leq 4$. *Moreover,* $\chi_\rho(C_n) = 3$ *if and only if* $n = 3$ *or* $n \equiv 0 \pmod 4$.

In recent years, the packing coloring of cubic graphs was interested several authors, in 2016, Gastineau and Togni [**?**] constructed a cubic graph G with $\chi_\rho(G) = 13$, and asked whether there are cubic graphs with $\chi_\rho(G) > 13$. In the same year Brešar *et al.* [**?**] answered this question in affirmative by constructing a cubic graph G with $\chi_\rho(G) = 14$. Recently Balogh *et al.* [**?**] have proved that there are cubic graphs with arbitrarily large packing chromatic number.
In this conference, we will present in **??** the exact value of the i-packing nomber of generalized Peterson graphs and in Section **??** we determine some upper and lower bounds of the packing chromatic number of generalized Petersen graphs.

2. *I*-PACKING COLORING OF GENERALIZED PETERSON GRAPHS

In this section we determine the exacte value of the i-packing number, for all value of i, of generalized Peterson graphs.
The generalized Petersen graph, denoted by GP_n, is defined as follow : given an integer $n \geq 3$, it is defined by the vertices set of $2n$ vertices $V(GP_n) = V_1 \cup V_2$ such that $V_1 = \{x_i/1 \leq i \leq n\}$ and $V_2 = \{y_i/1 \leq i \leq n\}$. The edges set is given by $x_i x_{i+1}, x_i y_i, y_i y_{i+2}$ for all $1 \leq i \leq n$.

Obviously the generalized Petersen graph is one of the most famous objects in graph theories, and we can observed that this graph is cubic.

Starting by giving the exct value when $i = 3$.

Theorem 3 *Let G a generalized Petersen graph.*

$$\rho_3(G) = \left\lfloor \frac{n}{4} \right\rfloor.$$

In the other cases, we have the following result.

Theorem 4 *Let G a generalized Petersen graph. For an integer* $i \geq 4$, *we have :*

$$\rho_i(G) = \left\lfloor \frac{n}{2i-3} \right\rfloor.$$

3. PACKING COLORING OF GENERALIZED PETERSON GRAPHS

In this section, we are interested in the packing coloring problème.
In the following theorem we proof that the packing chromatic number of the generalized Petersen Graph, GP_n, with n is even, is between 7 and 12.

Theorem 5 *Let n an even integer and let GP_n be a generalized Petersen graph. We then have*

$$7 \leq \chi_\rho(GP_n) \leq 12$$

and these two bounds are tight.

For the particular case when $n = 5$, we have the following reult.

Proposition 6 *For Petersen graph GP_5, we have $\chi_\rho(GP_5) = 6$.*

We complete the study of packing coloring of generalized peterson graphs by this theorem.

Theorem 7 *Let n an odd integer and $n \geq 7$, and let GP_n be a generalized Petersen graph. We then have*

$$7 \leq \chi_\rho(GP_n) \leq 15.$$

4. CONCLUSIONS

In this work, we have studied the packing coloring of generalized Peterson graphs. We have determined the exact value of – or upper and lower bounds on – the packing chromatic number of this class of graphs. We leave as open problems the following questions.

1. What is the relation betwwen the packing coloring problem and the i-distance coloring of generalized Peterson Grpahs ?
2. What is the best possible upper and lower bounds on the packing chromatic number of generalized Peterson graphs ?

5. REFERENCES

[1] G. Argiroffo, G. Nasini and P. Torres, Polynomial instances of the packing coloring problem, *Electron. Notes Discrete Math.* 37 (2011), 363–368.

[2] G. Argiroffo, G. Nasini and P. Torres, The packing coloring problem for $(q, q-4)$-graphs, *Lecture Notes in Comput. Sci.* 7422 (2012), 309–319.

[3] G. Argiroffo, G. Nasini and P. Torres, The packing coloring problem for lobsters and partner limited graphs, *Discrete Appl. Math.* 164 (2014), 373–382.

[4] J. Balogh, A. Kostochka and X. Liu. Packing chromatic number of subcubic graphs, *arXiv :1703.09873v1* [math.CO] 29 Mar 2017.

[5] B. Brešar, S. Klavžar and D.F. Rall, On the packing chromatic number of Cartesian products, hexagonal lattice, and trees, *Discrete Appl. Math.*, 155 (2007), 2303–2311.

[6] B. Brešar, S. Klavžar, D.F. Rall and K. Wash. Packing Chromatic number under local changes in a graph, *Discrete Math.*, 340 (2017), 11101115.

[7] J. Ekstein, J. Fiala, P. Holub and B. Lidický, The packing chromatic number of the square lattice is at least 12, March 12, 2010, arXiv :1003.2291v1 [cs.DM].

[8] J. Ekstein, P. Holub and B. Lidický, Packing chromatic number of distance graphs, *Discrete Appl. Math.* 160 (2012), 518–524.

[9] J. Ekstein, P. Holub and O. Togni, The packing coloring of distance graphs $D(k,t)$, *Discrete Appl. Math.* 167 (2014), 100–106.

[10] J. Fiala and P. A. Golovach, Complexity of the packing coloring problem for trees, *Discrete Appl. Math.* 158 (2010), 771–778.

[11] J. Fiala, S. Klavžar and B. Lidický, The packing chromatic number of infinite product graphs, *European J. Combin.* 30 (2009), 1101–1113.

[12] A.S. Finbow and D. F. Rall, On the packing chromatic number of some lattices, *Discrete Appl. Math.* 158 (2010), 1224–1228.

[13] N. Gastineau and O. Togni. $S-$packing colorings of cubic graphs. *arXiv : 1403.7495v2 [cs.DM]*,

[14] W. Goddard, S.M. Hedetniemi, S.T. Hedetniemi, J.M. Harris and D.F. Rall, Broadcast chromatic numbers of graphs, *Ars Combin.* 86 (2008), 33–49.

[15] D. Korže and A. Vesel, On the packing chromatic number of square and hexagonal lattice, *Ars Math. Contemp.* 7 (2014), 13–22.

[16] D. Laïche, I. Bouchemakh and É. Sopena. On the Packing Coloring of Undirected and Oriented Generalized Theta Graphs. *Australasian Journal of Combinatorics*, 66 (2) : 310-329, 2016.

[17] D. Laïche, I. Bouchemakh and É. Sopena. Packing coloring of some undirected and oriented coronae graphs. *Discussiones Mathematicae Graph Theory*, accepted, 2016.

[18] D.F. Rall, B. Brešar, A.S. Finbow and S. Klavžar, On the packing chromatic number of trees, Cartesian products and some infinite graphs, *Electron. Notes Discrete Math.* 30 (2008), 57–61.

[19] R. Soukal and P. Holub, A note on the packing chromatic number of the square lattice, *Electron. J. Combin.* 17 (2010).

[20] O. Togni, On packing colorings of distance graphs, *Discrete Appl. Math.* 167 (2014), 280–289.

[21] P. Torres and M. Valencia-Pabon, The packing chromatic number of hypercubes, *Discrete Appl. Math.* 190-191 (2015), 127–140.

[22] A. William, I. Rajasingh and S. Roy, Packing chromatic number of enhanced hypercubes, *Int. J. Math. Appl.* 2 (2014), no. 3, 1–6.

[23] A. William and S. Roy, Packing chromatic number of certain graphs, *Int. J. Pure Appl. Math.* 87 (2013), no. 6, 731–739.