FIRST ORDER EVOLUTION INCLUSIONS GOVERNED BY SWEEPING PROCESS IN BANACH SPACES

F. Selamnia and D. Azzam-Laouir
Department of Mathematics, Jijel university

ABSTRACT

In this paper, we prove the existence of solutions for the sweeping process in separable Banach space of the form

$$-\dot{u}(t)) \in N_{C(t,u(t))}(u(t)) + F(t,u(t)),\, a.e. t \in [0,T].$$

where F is an upper semicontinuous set-valued mapping with nonempty closed convex values. C a nonempty ball compact and r-prox-regular E set- valued mapping and $N_{C(t,u)}(.)$ is the proximal normal cone of C(t,u).

1. INTRODUCTION

The aim of this paper is to prove existence results of solutions for the first order differential inclusion in a separable Banach space E of the form

$$(\mathscr{P}_F)\begin{cases} u(0) = u_0; \\ -\dot{u}(t)) \in N_{C(t,u(t))}(u(t)) + F(t,u(t)), \;\; a.e.\, t \in [0,T]. \end{cases}$$

where $C : [0,T] \rightrightarrows E$ is a non empty ball compact and r-prox-regular valued multifunction, $N_{C(t,u)}(.)$ is the proximal normal cone of C(t,u) and $F : [0,T] \times E \longrightarrow E$ is an upper semicontinuous set-valued mapping with nonempty closed convex values.

Then the main concept, which appeared to get around the convexity of sets C(t), is the notion of uniform prox-regularity. This property is very well-adapted to the resolution of (1) : a set C is said to be η-prox-regular if the projection on C is single valued and continuous at any point whose the distance to C is smaller than η.

Numerous works have been devoted to applications of prox-regularity in the study of sweeping process. The case without perturbation (F = 0) was firstly treated by G. Colombo, V.V. Goncharov by H. Benabdellah and later by L. Thibault and by G. Colombo, M.D.P. Monteiro Marques. by L. Thibault the considered problem is

$$(1)\begin{cases} -du \in N(C(t),u(t)) \\ u(T_0) = u_0, \end{cases}$$

where du is the differential measure of u. The existence and uniqueness of solutions of (5) are proved with similar assumptions as previously. In infinite dimension with assuming that E is a Hilbert space $E = H$, the perturbed problem is studied by M. Bounkhel, J.F. Edmond and L. Thibault. For example L. Thibault show the well-poshness of

$$(2)\begin{cases} -du \in N(C(t),u(t)) + F(t,u(t))dt \\ u(0) = u_0, \end{cases}$$

with a set-valued map C taking $\eta-$prox regular values (for some $\eta > 0$).
Indeed the main difficulty of this problem is the weak smoothness of the proximal normal cone. For a fixed closed subset C, the set-valued map $x \longrightarrow N(C,x)$ is not upper semi-continuous, which is needed for the proof. The prox-regularity implies this required smoothness. We finish by presenting the work of H. Benabdellah. He deals with sweeping process in an abstract Banach framework, in considering the Clarke normal cone, which satisfies this upper semi-continuity.
Since then, important improvements have been developed by weakening the assumptions in order to obtain the most general result of existence for sweeping process. Adopted by us in our work on this without the work of the researchers F. Bernicot and J. Venel is to prove existence results for sweeping process associated to a moving set $t \mapsto C(t)$ on a time interval $I := [0,T]$ in considering the proximal normal cone (which will be denoted N). Let E be a Banach space and $C : I \to E$ be a set-valued map with nonempty closed values, and let $F : I \times E \rightrightarrows E$ be a set-valued map taking nonempty convex compact values. An associated sweeping process $u : I \to E$ is a solution of the following differential inclusion

$$(3) \begin{cases} u(0) = u_0; \\ u(t) \in C, \ \forall t \in [0,T] \\ \frac{du(t)}{dt} + N(C(t), u(t)) \in F(t, u(t)), \ a.e. \ t \in [0,T]. \end{cases}$$

This differential inclusion can be thought as following : the point $u(t)$, submitted to the field $F(t,u(t))$, has to live in the set C(t) and so follows its time-evolution.
In a Hilbert space H the authors proved the existence of solutions of the perturbed sweeping process,

$$(4) \begin{cases} u(0) = u_0 \in C(0, u_0); \\ -\dot{u}(t) \in N(C(t, u(t)), u(t)), \ a.e. \ t \in [0,T]. \end{cases}$$

where $C : [0,T] \times H \rightrightarrows H$ is a nonempty, closed, and convex valued multifunction.

2. NOTATION AND PRELIMINARIES

Let $\mathbf{C}_E([0,1])$ be the Banach space of all continuous mappings
$u : [0,1] \to E$, endowed with the sup-norm, and $\mathbf{C}_E^1([0,1])$ be the Banach space of all continuous mappings $u : [0,1] \to E$ with continuous derivative, equipped with the norm

$$\|u\|_{\mathbf{C}_E^1} = \max\{\max_{t\in[0,1]} \|u(.)\|, \max_{t\in[0,1]} \|\dot{u}(.)\|\}.$$

For $A \in E, co(A)$ denotes the convex hull of A and $co(A)$ its closed convex hull. We have the following characterization

Definition 1 *For a subset A of E and a point $x \in E$, the distance between x and K, denoted by d(x,A), is defined as the smallest distance from x to element of A. More presley*

$$d(x,A) = \inf_{y\in A} d(x,y).$$

Definition 2 *Let A be a closed subset of E. Then the set-valued projection operator P_A is defined by*

$$\forall x \in E, \ \ P_A(x) = \{y \in E, \ \|x-y\| = d(x,A)\}.$$

Definition 3 *Let A be a closed subset of E and $x \in A$, we denote by $N_A(x)$ the proximal normal cone of A at x, defined by*

$$N_A(x) = \{v \in E, \exists s > 0, \ x \in P_A(x+sv)\}.$$

Proposition 1 *Let $r \in (0,+\infty]$, let A be a nonempty closed and uniformly r-prox-regular subset of E. Then we have the following*

- *For all $x \in E$ with $d(x,A) < r$, the projection of x onto A is well-defined and continuous, that is, $P_A(x)$ is single-valued;*
- *The sub differential proximal the $d(.,A)(\partial^P d(x,A))$ coincides with all sub differentials contained in the sub Clarke differential $(\partial^C d(x,A))$ at any point $x \in E$ satisfying $d(x,A) < r$.*
- *For all $r' \in (0,r)$, the projection operator is Lipschitz on the set of points $x \in E$ satisfying $d(x,A) < r'$, with Lipschitz constant*

$$\frac{r}{r-r'}$$

- *if $u = P_A(x)$ then, $u = P_A(u + r\frac{x-u}{\|x-u\|})$.*

In the sequel we will note by $\partial d(A,x)$ the sub differential of the distance function associated with nonempty closed and uniformly r-prox-regular subset of E (without specifying the name because they are all equal).

Definition 4 *The Banach space E is said to be strictly convex if*

$$x,y \in S \ with \ x \neq y \Longrightarrow \|(1-\lambda)x + \lambda y\| < 1 \ for \ \lambda \in (0,1).$$

This means that the unit sphere S contains no line segments.

Definition 5 *Let E be a Banach space, equipped with its norm $\|.\|_E$.*

(i) The space E is said to be uniformly convex if for all $\varepsilon > 0$, there is some $\delta > 0$ so that for any two vectors $x,y \in E$ with $\|x\|_E \leq 1$ and $\|y\|_E \leq 1$ we have

$$\|x+y\|_E > 2 \Longrightarrow \|x-y\|_E \leq \varepsilon.$$

(ii) The space E is said to be uniformly smooth if his norm is uniformly Freshet differentiable away of 0, it means that for any two unit vectors $x_0, h \in E$, the limit

$$\lim_{t\to 0} \frac{\|x_0 + th\|_E - \|x_0\|_E}{t}$$

exists uniformly with respect to $h, x_0 \in S(0,1)$.

Lemma 1 *Let E be a Banach space and K be a closed subset of E. Then for $x \in A$ and $v \in \Gamma^r(A,x)$, we have $\lambda v \in \Gamma^r(A,x)$ for all $\lambda \in (0,1)$. Therefore, if we assume that E is uniformly convex then for all $\lambda \in (0,1)$, we have $x \in P_A(x + \lambda r v)$.*

The first part is well-known, the second part is quite more complicated.
We recall a famous result (due to D. Milman and B.J. Pettis)

Theorem 1 *If a Banach space E is uniformly convex then it is reflexive.*

Theorem 2 *If E' is uniformly convex then E is uniformly smooth and E is reflexive. If E' is separable then E is separable.*

Now we consider some results concerning the smoothness of the norm.

Remark 1 *If E is uniformly smooth, then $x \longmapsto \|x\|_E$ is C^1 on $E \setminus \{0\}$.*

Proposition 2 *If E is uniformly smooth, then for all $x \in E \setminus \{0\}$, we have*

$$\langle (\nabla \parallel . \parallel_E)(x), x \rangle = \|x\|_E.$$

By triangle inequality, $\|(\nabla \|.\|_E)(x)\|_{E'} = 1$.

Now, as we know that the norm could be non differentiable at the origin 0, we study the function $x \longmapsto \|x\|_E^p$ for an exponent $p > 1$.

Proposition 3 *Let E be a uniformly smooth Banach space and $p \in (1, \infty)$ be an exponent. The function $x \longmapsto \|x\|_E^p$ is C^1 over the whole space E.*

Definition 6 *For E a uniformly smooth Banach space and $p \in (1, \infty)$, we denote*

$$J_p(x) := \frac{1}{p}(\nabla \|.\|_E^p)(x) \in E'.$$

Definition 7 *A subset $A \subset E$ is said to be ball-compact if for all closed ball $\overline{\mathbf{B}} = \overline{\mathbf{B}}(x, R)$ of E, the set $\overline{\mathbf{B}} \cap A$ is compact.*
Obviously a ball-compact subset A is closed.

Definition 8 *Let I be an interval of $\mathbb{R}$. A separable reflexive uniformly smooth Banach space E is said to be $I-$smoothly weakly compact" for an exponent $p \in (1, \infty)$ if for all bounded sequence $(x_n)_{n \geq 0}$ of $L^\infty(I, E)$, we can extract a subsequence $(y_n)_{n \geq 0}$ weakly converging to a point $y \in L^\infty(I, E)$ such that for all $z \in L^\infty(I, E)$ and $\phi \in L^1(I, \mathbb{R})$,*

$$\lim_{n \to \infty} \int_I \underset{E'}{\langle} J_p(z(t) + y_n(t)) - J_p(y_n(t)), y_n(t) \rangle_E \phi(t) dt = \int_I \underset{E'}{\langle} J_p(z(t) + y(t)) - J_p(y(t)), y(t) \rangle_E \phi(t) dt. \quad (2.1)$$

3. MAIN RESULT

Now, we are able to prove our main existence theorem.

Theorem 3 *Let $I = [0, T]$ $(T > 0)$ and E be a separable, reflexive, uniformly smooth Banach space, which is I-smoothly weakly compact for an exponent $p \in [2, \infty)$. Let $F : I \times E \longrightarrow E$ be an upper semicontinuous set-valued mapping with nonempty closed convex values. We assume that there exists a constant $m > 0$ such that*

$$F(t, x) \subset m\overline{B}_E, \quad \forall (t, x) \in I \times E. \qquad (3.1)$$

Let $r > 0$ and $C : [0, T] \times E \longrightarrow E$ be a set-valued mapping taking nonempty ball-compact and r-prox-regular values. We assume that there exists $l_1 \geq 0, 0 \leq l_2 < 1$ such that

$$H(C(t, u), C(s, v)) \leq l_1 |t - s| + l_2 \|u - v\|, \quad \forall s, t \in I and u, v \in E \qquad (3.2)$$

Then for all $u_0 \in C(0, u_0)$ the differential inclusion

$$(\mathscr{P}_r) \begin{cases} u(0) = u_0; \\ u(t) \in C(t, u(t)), \quad \forall t \in I \\ -\dot{u}(t)) \in N_{C(t,u(t))}(u(t)) + F(t, u(t)), \ \ a.e.\ t \in [0, T]. \end{cases}$$

has a Lipschitz solution $u : I \longrightarrow E$.

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