ON NONHOMOGENEOUS P-LAPLACIAN ELLIPTIC EQUATIONS INVOLVING A CRITICAL SOBOLEV EXPONENT AND MULTIPLE HARDY-TYPE TERMS

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ABSTRACT

In this paper we consider a class of nonhomogeneous p-Laplacian elliptic equations with a critical Sobolev exponent and multiple Hardy type terms. By Ekeland variational principale on Nehari manifold and mountain pass lemma, we prove the existence of multiple solutions under sufficient conditions on the data and the considered parameters.

1. INTRODUCTION

In this paper we study the existence and multiplicity of positive solutions for the quasilinear elliptic problem :

$$(\mathscr{P})\begin{cases} -\Delta_p u - \sum_{i=1}^{k} \dfrac{\mu_i}{|x-a_i|^p}\,|u|^{p-2}\,u = |u|^{p^*-2}\,u + \sum_{i=1}^{k} \dfrac{\lambda_i}{|x-a_i|^{p-s_i}}\,|u|^{p-2}\,u + f & \text{in } \Omega \\ u = 0 & \text{on } \partial\,\Omega, \end{cases}$$

where Ω is an open smooth bounded domain of $\mathbb{R}^N (N \geq 3), 1 < p < N$, $k \in \mathbb{N}^*$, $a_i \in \Omega$, λ_i and μ_i are nonnegative parameters and s_i are positive constants $(1 \leq i \leq k)$; f is a bounded measurable function which is positive in each neighborhood of a_i. Here $p^* = \frac{pN}{N-p}$ denotes the critical Sobolev exponent and $\Delta_p u = div(|\nabla u|^{p-2}\,\nabla u)$ is the p-Laplacian operator.

Problem $(\mathscr{P})$ is related to the Hardy inequality [7] :

$$\int_\Omega \frac{|u|^p}{|x-a|^p}\,dx \leq \frac{1}{\overline{\mu}} \int_\Omega |\nabla u|^p\,dx, \text{ for all } u \in C_0^\infty(\Omega), \tag{1}$$

where $a \in \Omega$ and $\overline{\mu} = \left(\frac{N-p}{p}\right)^p$ is the best Hardy constant. We shall work with the space $W = W_0^{1,p}(\Omega)$ the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\| := \left(\int_\Omega \left(|\nabla u|^p - \sum_{i=1}^{k} \frac{\mu_i}{|x-a_i|^p}\,|u|^p\right) dx\right)^{1/p},$$

with $1 < p < N$, $\mu_i > 0$ for $i = 1,...,k$ and $\sum_{i=1}^{k} \mu_i < \overline{\mu}$. In particular, Hardy's inequality shows that this norm is equivalent to the usual norm $(\int_\Omega |\nabla u|^p\,dx)^{1/p}$.

Many research works related to problem $(\mathscr{P})$ were considered by some authors in recent years. We mention especially the interesting works of :

-Abdellaoui et. al. [1] studied the following problem :

$$-\Delta_p u = \frac{\lambda h(x)}{|x|^p} |u|^{q-1} u + g(x) |u|^{p^*-1} u \text{ in } \mathbb{R}^N,$$

where h and g are two bounded measurable functions, they proved the existence and nonexistence results for two cases, they first proved for the equation with a concave singular term, then they studied the critical case related to hardy inequality, providing a description of the behavior of radiale solutions of the limiting problem and obtaining existence and multiplicity results for perturbed problems through variational and topological arguments.

-Haidong Liu proved in [8] the existence of two solutions of the following problem :

$$\begin{cases} -\Delta_p u = \mu V(x) |u|^{p-2} u + |u|^{p^*-2} u + \lambda f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\,\Omega, \end{cases}$$

under some sufficient assumptions on V, f, λ and μ, where $V(x)$ is a linear weight and f is a positive function. The case $p = 2$ has been treated by Chen [3], who proved the existence of at least m positive solutions.

-Hsu studied in [10] the existence and multiplicity of positive solutions of the quasilinear elliptic problem :

$$\begin{cases} -\Delta_p u - \sum_{i=1}^{k} \dfrac{\mu_i}{|x-a_i|^p} |u|^{p-2} u = |u|^{p^*-2} u + \lambda\, |u|^{q-2} u & \text{in } \Omega \\ u = 0 & \text{on } \partial\,\Omega, \end{cases}$$

using Nehari manifold and mountain pass lemma he prove the existence of two solutions for $1 \leq q < p$ and some assumptions on the parameters μ_i, λ.

Remark 1 *The case $p = 2$ in the problem considered $(\mathscr{P})$ has been treated in [2].*

To state our results, we need some notions.

Let A_i, B_i $(A_i < B_i)$ be the zeroes of the function $g(t) = (p-1)t^p - (N-p)t^{p-1} + \mu_i,\ t \geq 0$ (for $p = 2$ we have $A_i = \sqrt{\overline{\mu}} - \sqrt{\overline{\mu} - \mu_i}$, $B_i = \sqrt{\overline{\mu}} + \sqrt{\overline{\mu} - \mu_i}$), $1 \leq i \leq k$.

Let us denote

$$s_i^* = p(1 + B_i) - N$$

$$\lambda^* := \min_{j=1,..,k} \left\{ \lambda_1 \left(s_j \right) \right\},$$

where

$$\lambda_1 \left(s_j \right) := \inf_{u \in W \setminus \{0\}} \left\{ \|u\|^p : \int_\Omega \frac{|u|^p}{\left|x - a_j\right|^{p-s_j}} dx = 1 \right\},$$

with $1 < p < N$ and $s_j > 0, 1 \leq j \leq k$.

Now, we consider the following hypothesis :

$(\mathscr{H}1)$ f is positive function in each neighborhood of a_i and satisfies

$$\int_\Omega f u\, dx < C_p \left(\|u\|^p - \sum_{i=1}^{k} \lambda_i \int_\Omega \frac{|u|^p}{|x-a_i|^{p-s_i}} dx \right)^{\frac{p^*-1}{p^*-p}}$$

for all $u \in W$ such that $\int_\Omega |u|^{p^*} dx = 1$ and $C_p = \left(\frac{p^*-p}{p-1}\right) \left(\frac{p-1}{p^*-1}\right)^{(p^*-1)/(p^*-p)}$.

$(\mathscr{H}2)$ We consider $\varepsilon > 0$ small enough, $\delta = (N-p)/p$ and $1 \leq l \leq k$ such that $\int_\Omega f u_{\varepsilon,i}\, dx = O\left(\varepsilon^\theta \left|\ln(\varepsilon)\right|\right)$ with $\theta < \min(B_l - \delta, \delta - A_l)$ and $u_{\varepsilon,i} \in W$.

Remark 2 *If $g \in L^q(\Omega)$ is a positive function with $q = p^*/(p^*-1)$ and*

$$\left(\int_\Omega g^q dx\right)^{\frac{1}{q}} < C_p \left[\frac{\lambda^* - \sum_{i=1}^k \lambda_i}{\lambda^*(p^*-1)}\right]^{\frac{p(p^*-1)}{p^*-p}} S^{\frac{p^*-1}{p^*-p}}$$

then g satisfies $(\mathscr{H}1)$. Moreover, if $f(x) = \varepsilon e^{|\ln \varepsilon^2|} g(x)$ for $\varepsilon > 0$ small enough, then $f \in L^q(\Omega)$ satisfies $(\mathscr{H}1)$ and $(\mathscr{H}2)$.

The main result of this paper is the following theorem.

Theorem 1 *Assume that $\mu_i \geq 0$, $\lambda_i \geq 0$, $s_i > 0$, $\sum_{i=1}^k \mu_i < \overline{\mu}$, $\sum_{i=1}^k \lambda_i < \lambda^*$ and f satisfies $(\mathscr{H}1)$ and $(\mathscr{H}2)$. Then the problem $(\mathscr{P})$ has at least $2k$ solutions in W.*

2. PRELIMINARY RESULTS

We give here some results which play important roles in the sequel of this work.

In what follows we denote the norms of $L^q(\Omega)$, $(1 \leq q < \infty)$ and W^{-1}(the dual of W) by $|u|_q$ and $\|u\|_{-}$ respectively. $L^p(\Omega, |x-a_i|^s)$ denotes the usual weighted $L^p(\Omega)$ space with the weight $|x-a_i|^s$. C, C_i will denote various positive constants whose exact values are not important. By $B_{a_j}^r$ we denote the open ball in Ω with center at a_j and radius $r > 0$.

We define for $\mu_i \in (0, \overline{\mu})$ and $a_i \in \Omega$ the constant :

$$S_{\mu_i}(\Omega) := \inf_{u \in W \setminus \{0\}} \frac{\int_\Omega \left(|\nabla u|^p - \mu_i \frac{|u|^p}{|x-a_i|^p}\right) dx}{|u|_{p^*}^p}, 1 \leq i \leq k.$$

From [5], S_{μ_i} is independent of any $\Omega \subset \mathbb{R}^N$ in the sense that $S_{\mu_i}(\Omega) = S_{\mu_i}(\mathbb{R}^N) = S_{\mu_i}$. In addition, the constant S_{μ_i} is achieved by a family of functions

$$V_{\varepsilon,i}(x) := \varepsilon^{(p-N)/p} U_i\left(\frac{x-a_i}{\varepsilon}\right)$$

where the positive radial function U_i is defined in [1] and $\varepsilon > 0$. Moreover, the function $V_{\varepsilon,i}$ satisfies :

$$\begin{cases} -\Delta_p V_{\varepsilon,i} - \mu_i \dfrac{|V_{\varepsilon,i}|^{p-1} V_{i,\varepsilon}}{|x-a_i|^p} = |V_{\varepsilon,i}|^{p^*-2} V_{\varepsilon,i} & \text{in } \mathbb{R}^N \setminus \{a_i\} \\ u \longrightarrow 0 & \text{as } |x| \longrightarrow \infty. \end{cases}$$

Now, we shall give some estimates for the extremal functions $V_{\varepsilon,i}$ which we will use later. Let $\varphi_i \in C_0^\infty(\Omega)$ such that

$$0 \leq \varphi_i(x) \leq 1, \varphi_i(x) = \begin{cases} 0 & \text{if } |x-a_i| \geq 2r \\ 1 & \text{if } |x-a_i| \leq r \end{cases} ; \text{ and } |\nabla \varphi_i(x)| \leq C$$

where δ is a small positive number. Put $u_{\varepsilon,i} = \varphi_i(x) V_{\varepsilon,i}(x)$ for i in $\{1, ..., k\}$.

In what follows, we consider $s_i, \lambda_i > 0$ and $\mu_i \geq 0$ such that $\sum_{i=1}^k \mu_i < \overline{\mu}$ and $\sum_{i=1}^k \lambda_i < \lambda^*$.

By [6], we have the following estimates.

Lemma 2 *Assume that $v \in W$ is a positive solution of problem $(\mathscr{P})$ and $1 < p < N$, then for $\varepsilon > 0$ small enough and $\delta = (N-p)/p$, we have*

$$\begin{aligned}
&\int_\Omega \left(\left|\nabla u_{\varepsilon,i}\right|^p - \frac{\mu_i}{|x-a_i|^p} \left|u_{\varepsilon,i}\right|^p \right) dx = S_{\mu_i}^{N/p} + \mathscr{O}\left(\varepsilon^{p(B_i-\delta)}\right),\\
&\int_\Omega \left|u_{\varepsilon,i}\right|^{p^*} dx = S_{\mu_i}^{N/p} - \mathscr{O}\left(\varepsilon^{p^*(B_i-\delta)}\right),\\
&\int_\Omega |v| \left|u_{\varepsilon,i}\right|^{p^*-1} dx = \mathscr{O}\left(\varepsilon^{(\delta-A_i)}\right),\\
&\int_\Omega \left|u_{\varepsilon,i}\right| |v|^{p^*-1} dx = \mathscr{O}\left(\varepsilon^{(p^*-1)(\delta-A_i)}\right),
\end{aligned}$$

$$\int_\Omega \left|\nabla u_{\varepsilon,i}\right|^{p-1} |\nabla v| \, dx = \begin{cases} \mathscr{O}\left(\varepsilon^{(\delta-A_i)}\right) & , \ A_i + (p-1) B_i > p\delta \\ \mathscr{O}\left(\varepsilon^{(\delta-A_i)} |\ln(\varepsilon)|\right) & , \ A_i + (p-1) B_i = p\delta \\ \mathscr{O}\left(\varepsilon^{(p-1)(B_i-\delta)}\right) & , \ A_i + (p-1) B_i < p\delta \end{cases}$$

$$\int_\Omega |\nabla v|^{p-1} \left|\nabla u_{\varepsilon,i}\right| dx = \begin{cases} \mathscr{O}\left(\varepsilon^{(p-1)(\delta-A_i)}\right) & , \ B_i + (p-1) A_i > p\delta \\ \mathscr{O}\left(\varepsilon^{(B_i-\delta)} |\ln(\varepsilon)|\right) & , \ B_i + (p-1) A_i = p\delta \\ \mathscr{O}\left(\varepsilon^{(B_i-\delta)}\right) & , \ B_i + (p-1) A_i < p\delta \end{cases}$$

$$\int_\Omega \frac{\left|u_{\varepsilon,i}\right|^{p-1} |v|}{|x-a_i|^p} dx = \begin{cases} \mathscr{O}\left(\varepsilon^{(\delta-A_i)}\right) & , \ (p-1) B_i + A_i > p\delta \\ \mathscr{O}\left(\varepsilon^{(p-1)(B_i-\delta)} |\ln(\varepsilon)|\right) & , \ (p-1) B_i + A_i = p\delta \\ \mathscr{O}\left(\varepsilon^{(p-1)(B_i-\delta)}\right) & , \ (p-1) B_i + A_i < p\delta \end{cases}$$

and

$$\int_\Omega \frac{|v|^{p-1} \left|u_{\varepsilon,i}\right|}{|x-a_i|^p} dx = \begin{cases} \mathscr{O}\left(\varepsilon^{(p-1)(\delta-A_i)}\right) & , \ B_i + (p-1) A_i > p\delta \\ \mathscr{O}\left(\varepsilon^{Bi-\delta} |\ln(\varepsilon)|\right) & , \ B_i + (p-1) A_i = p\delta \\ \mathscr{O}\left(\varepsilon^{(B_i-\delta)}\right) & , \ B_i + (p-1) A_i < p\delta \end{cases}$$

Let

$$I(u) := \int_\Omega \left(|\nabla u|^p - \sum_{i=1}^k \mu_i \frac{|u|^p}{|x-a_i|^p} - \sum_{i=1}^k \lambda_i \frac{|u|^p}{|x-a_i|^{p-s_i}} \right) dx,$$

$$S^* := \inf_{u \in W \setminus \{0\}} \left\{ (I(u))^{1/p} ; | \, |u|_{p^*} = 1 \right\}.$$

From the fact that $\sum_{i=1}^k \lambda_i < \lambda^*$, we have $S^* > 0$.

The energy functional associated to $(\mathscr{P})$ is given by the following expression :

$$J(u) := \frac{1}{p} I(u) - \frac{1}{p^*} \int_\Omega |u|^{p^*} dx - \int_\Omega f u dx.$$

We see that J is well defined in W and belongs to $C^1(W, \mathbb{R})$.

It is known that a weak solution $u \in W$ of $(\mathscr{P})$ corresponds to a critical point of J which is given by :

$$\langle J'(u), \varphi \rangle = \int_\Omega \left(|\nabla u|^{p-2} \nabla u \nabla \varphi - \sum_{i=1}^k \frac{\mu_i |u|^{p-2}}{|x-a_i|^p} u\varphi - \sum_{i=1}^k \frac{\lambda_i |u|^{p-2}}{|x-a_i|^{p-s_i}} u\varphi \right) dx$$

$$- \int_\Omega |u|^{p^*-2} u \varphi dx - \int_\Omega f \varphi dx = 0, \qquad \text{for all } \varphi \in W.$$

More standard elliptic regularity argument implies that a weak solution $u \in W$ is indeed in $C^2(\Omega \backslash \{a_1, a_2, ..., a_k\}) \cap C^1(\overline{\Omega} \backslash \{a_1, a_2, ..., a_k\})$ and we can say that u satisfies $(\mathscr{P})$ in the classical sense.

Definition 1 *A functional $J \in C^1(W, \mathbb{R})$ satisfies the Palais–Smale condition at level c, ($(PS)_c$ for short), if any sequence $(u_n) \subset W$ such that*

$$J(u_n) \longrightarrow c \quad \text{and} \quad J'(u_n) \longrightarrow 0 \quad \text{in} \quad W^{-1} \text{ (dual of } W\text{)},$$

contains a strongly convergent subsequence.

As J is not bounded below on W, it is useful to consider it on the Nehari manifold :

$$\mathscr{N} = \left\{u \in W \backslash \{0\} : \left\langle J'(u), u \right\rangle = 0\right\}.$$

It is natural to split $\mathscr{N}$ into three subsets :

$$\mathscr{N}^{+} = \{u \in \mathscr{N} : \langle J''(u), u \rangle > 0\},$$

$$\mathscr{N}^{-} = \{u \in \mathscr{N} : \langle J''(u), u \rangle < 0\},$$

$$\mathscr{N}^{0} = \{u \in \mathscr{N} : \langle J''(u), u \rangle = 0\},$$

with

$$\begin{aligned} \left\langle J''(u), u \right\rangle &= pI(u) - p^* |u|_{p^*}^{p^*} - \int_\Omega fu\, dx \\ &= (p-1) I(u) - (p^*-1) |u|_{p^*}^{p^*} \\ &= (p-p^*) I(u) + (p^*-1) \int_\Omega fu\, dx. \end{aligned}$$

Lemma 3 *Let f satisfies the condition $(\mathscr{H}1)$. Then for any $u \in W \backslash \{0\}$ there exists an unique $t^+ = t^+(u) > 0$ such that $t^+ u \in \mathscr{N}^-$ and*

$$t^+ > \left(\frac{(p-1) I(u)}{(p^*-1) |u|_{p^*}^{p^*}} \right)^{(p^*-1)/(p^*-p)} := t_{\max}(u) = t_{\max}$$

and $J\left(t^+ u\right) = \max_{t \geq t_{\max}} J(tu)$.

Moreover, if $\int_\Omega fu\, dx > 0$, then there exists an unique $t^- = t^-(u) > 0$ such that $t^- u \in \mathscr{N}^+$, $t^- < t_{\max}$ and $J\left(t^- u\right) = \inf_{0 \leq t \leq t_{\max}} J(tu)$.

Proof. The lemma is proved in the same way as in [13]. ∎

Lemma 4 *Let $f \neq 0$ satisfying the condition $(\mathscr{H}1)$ then $\mathscr{N}^0 = \varnothing$.*

Proof. Suppose that $\mathscr{N}^0 \neq \varnothing$. Then for $u \in \mathscr{N}^0$ we have :

$$(p-1) I(u) = (p^*-1) |u|_{p^*}^{p^*},$$

thus

$$\begin{aligned} 0 &= I(u) - |u|_{p^*}^{p^*} - \int_\Omega fu\, dx \\ &= (p^*-p) |u|_{p^*}^{p^*} - (p-1) \int_\Omega fu\, dx. \end{aligned} \tag{3}$$

From $(\mathscr{H}1)$ and (2.2) we obtain

$$\begin{aligned} 0 &< C_p\,(I(u))^{(p^*-1)/(p^*-p)} - \int_\Omega fu\,dx \\ &= (p^*-p)\,|u|_{p^*}^{p^*}\left[\left(\frac{(p-1)\,I(u)}{(p^*-1)\,|u|_{p^*}^{p^*}}\right)^{(p^*-1)/(p^*-p)} - 1\right] = 0, \end{aligned}$$

which yields to a contradiction. ■

Define, for $i \in \{1,..,k\}$,

$$\beta_i(u) := \frac{\int_\Omega \psi_i(x)\,|\nabla u|^p\,dx}{|\nabla u|_p^p}, \text{ where } \psi_i(x) = \min\{\rho, |x-a_i|\} \text{ and } \rho > 0.$$

Take $r_0 = \frac{\rho}{3}$ with $\rho < \frac{1}{4}\min_{i\neq j}|a_i - a_j|$ and let

$$\mathscr{N}_i^+ = \left\{u \in \mathscr{N}^+ : \beta_i(u) \le r_0\right\} \text{ and } \mathscr{N}_i^- = \left\{u \in \mathscr{N}^- : \beta_i(u) \le r_0\right\}.$$

Denote

$$m_i^+ := \inf_{u\in\mathscr{N}_i^+} J(u) \qquad \text{and} \qquad m_i^- := \inf_{u\in\mathscr{N}_i^-} J(u).$$

Lemma 5 ([3]) *Let $\rho > 0$ and r_0 defined as above. If $\beta_i(u) \le r_0$ then*

$$\int_\Omega |\nabla u|^p\,dx \ge 3\int_{\Omega\setminus B_i^\rho} |\nabla u|^p\,dx.$$

3. PROOF OF THEOREM 1

From now we consider j fixed in $\{1,...,k\}$.

3.1. Existence of solutions in $\mathscr{N}^+$

Using Ekeland's variational principle we prove the existence of k solutions in $\mathscr{N}^+$.

Proposition 6 *Let f be a bounded measurable function, locally positive in each neighborhood of a_i and satisfying satisfying $(\mathscr{H}1)$. Then $m_i^+ = \inf_{v\in\mathscr{N}_i^+} J(v)$ is achieved at a point $u_i \in \mathscr{N}_i^+$ which is a critical point and even a local minimum for J.*

Proof. We start by showing that J is bounded below in $\mathscr{N}$. Indeed, for $u \in \mathscr{N}^+$, we have

$$\frac{p-1}{p^*-1} I(u) > |u|_{p^*}^{p^*}.$$

The fact that $u \in \mathscr{N}$ we get :

$$\begin{aligned} J(u) &= \frac{1}{p} I(u) - \frac{1}{p^*}|u|_{p^*}^{p^*} - \int_\Omega fu\,dx \\ &= \left(\frac{1}{p} - \frac{1}{p^*}\right) I(u) - \left(1 - \frac{1}{p^*}\right)\int_\Omega fu\,dx \\ &\ge \left(\frac{1}{p} - \frac{1}{p^*}\right) I(u) - \left(1 - \frac{1}{p^*}\right)\|f\|_-\,\|u\| \\ &\ge \left(\frac{1-p}{pp^*}\right)\frac{(p^*-1)^{p/(p-1)}}{(p^*-p)^{1/(p-1)}}\,\|f\|_-^{p/(p-1)}. \end{aligned}$$

In particular

$$m_j^+ \geq m_0 \geq \left(\frac{1-p}{pp^*}\right)\frac{(p^*-1)^{p/(p-1)}}{(p^*-p)^{1/(p-1)}}\|f\|_-^{p/(p-1)},\text{ for } j=1,...,k$$

where $m_0 = \inf\limits_{u\in\mathcal{N}} J(u)$.

We claim that $m_j^+ < 0$. In fact, we know that $\int_{B_j^\varepsilon} fu_{\varepsilon,j} > 0$ for all ε small than a certain $\varepsilon_1 > 0$.

Set $0 < t_{\varepsilon,j}^- < t_{\varepsilon,j,\max}^-$ defined by Lemma 2 such that $t_{\varepsilon,j}^- u_{\varepsilon,j} \in \mathcal{N}^+$. From the fact that $\beta_j\left(t_{\varepsilon,j}^- u_{\varepsilon,j}\right)$ tends to 0 as ε goes to 0, it follows that there exists an ε_2 such that $\beta_j\left(t_{\varepsilon,j}^- u_{\varepsilon,j}\right) \leq r_0$ for $0<\varepsilon<\varepsilon_2<\varepsilon_1$.
Then $t_{\varepsilon,j}^- u_{\varepsilon,j} \in \mathcal{N}_j^+$, whence

$$\begin{aligned} J\left(t_{\varepsilon,j}^- u_{\varepsilon,j}\right) &= \frac{\left(t_{\varepsilon,j}^-\right)^p}{p} I(u_{\varepsilon,j}) - \frac{\left(t_{\varepsilon,j}^-\right)^{p^*}}{p^*}\left|u_{\varepsilon,j}\right|_{p^*}^{p^*} - t_{\varepsilon,j}^- \int_\Omega fu_{\varepsilon,j} \\ &= \left(\frac{1}{p}-1\right)\left(t_{\varepsilon,j}^-\right)^p I(u_{\varepsilon,j}) + \left(1-\frac{1}{p^*}\right)\left(t_{\varepsilon,j}^-\right)^{p^*}\left|u_{\varepsilon,j}\right|_{p^*}^{p^*} \\ &< \frac{(1-p)(p^*-p)}{pp^*}\left(t_{\varepsilon,j}^-\right)^p I(u_{\varepsilon,j}) < 0, \end{aligned}$$

this yields to $-\infty < m_0 \leq m_j^+ < 0$.

Ekeland's variational principle gives us a minimizing sequence $\left(u_{j,n}\right)_n \subset \mathcal{N}_j^+$ with the following properties :

$$\begin{aligned} &(i)\;\; J\left(u_{j,n}\right) < m_j^+ + \tfrac{1}{n} \\ &(ii)\;\; J(w) \geq J\left(u_{j,n}\right) - \tfrac{1}{n}\left|\nabla\left(w-u_{j,n}\right)\right|_p, \quad \text{for all } w\in\mathcal{N}_j^+. \end{aligned}$$

By taking n large, we have for some $\varepsilon \in (0,\varepsilon_2)$:

$$\begin{aligned} J\left(u_{j,n}\right) &= \left(\frac{1}{p}-\frac{1}{p^*}\right) I(u_{j,n}) - \left(1-\frac{1}{p^*}\right)\int_\Omega fu_{j,n}\,dx \\ &< m_j^+ + \frac{1}{n} \leq \frac{(1-p)(p^*-p)}{pp^*}\left(t_{\varepsilon,j}^-\right)^p I(u_{\varepsilon,j}). \end{aligned}$$

This implies

$$\int_\Omega fu_{j,n}\,dx \geq \frac{(p-1)(p^*-p)}{p(p^*-1)}\left(t_{\varepsilon,j}^-\right)^p I(u_{\varepsilon,j}) > 0.$$

Consequently, $u_{j,n} \neq 0$ and we get

$$\frac{(p-1)(p^*-p)}{p(p^*-1)}\left(t_{\varepsilon,j}^-\right)^p I(u_{\varepsilon,j}) \leq \left\|u_{j,n}\right\| \leq \frac{p^*-1}{p(p^*-p)}\|f\|_-.$$

Thus there exists a subsequence labeled $\left(u_{j,n}\right)_n$ such that $u_{j,n} \rightharpoonup u_j$ weakly in W, when n goes to $+\infty$. Using an argument similar to [13] we can conclude that $\left\|J'\left(u_{j,n}\right)\right\|_-$ tends to 0 as n goes to $+\infty$.

We deduce that

$$\left\langle J'\left(u_j\right),\varphi\right\rangle = 0,\text{ for all } \varphi\in W \tag{4}$$

i.e. u_j is a weak solution of $(\mathcal{P})$.

In particular $u_j \in \mathscr{N}$, and we have

$$\int_{\Omega} f u_j dx = \lim_{n \longrightarrow +\infty} \int_{\Omega} f u_{j,n} dx \geq \frac{(p-1)(p^* - p)}{p(p^* - 1)} \left(t_{\varepsilon,j}^-\right)^p I(u_{\varepsilon,j}) > 0.$$

Thus $u_j \neq 0$. Also, from Lemma 4 and (3.1) it follows that necessarily $u_j \in \mathscr{N}^+$.

By the fact that $\beta_j(u_j) = \lim_{n \longrightarrow \infty} \beta_j(u_{j,n}) \leq r_0$, then $u_j \in \mathscr{N}_j^+$. Hence

$$\begin{aligned} m_j^+ &\leq J(u_j) = \left(\frac{1}{p} - \frac{1}{p^*}\right) I(u_j) - \left(1 - \frac{1}{p^*}\right) \int_{\Omega} f u_j \, dx \\ &\leq \lim_{n \longrightarrow \infty} \inf J(u_{j,n}) = m_j^+. \end{aligned}$$

Hence, similarly to [13], we conclude that u_j is a local minimizer for J.

Then $u_{j,n} \longrightarrow u_j$ strongly in W and $J(u_j) = m_j^+ = \inf_{v \in \mathscr{N}_j^+} I(v)$. By Lemma 3, we deduce the existence of k solutions to the problem $(\mathscr{P})$. ■

3.2. Existence of solutions in $\mathscr{N}^-$

In this subsection, we shall find the range of c where J verifies the $(PS)_c$ condition.

Lemma 7 *If $c < \frac{1}{N} S_{\mu_l}^{N/p}$ where $S_{\mu_l}^{N/p} = \min\{S_{\mu_1}^{N/p}, ..., S_{\mu_k}^{N/p}, S_{\tilde{\lambda},\tilde{\mu}}^{N/p}\}$, then J satisfies the $(PS)_c$ condition.*

Proof. Let (u_n) be a $(PS)_c$ sequence for J with $c < \frac{1}{N} S_{\mu_l}^{N/p}$. We know that (u_n) is bounded in W, and there exist a subsequence of (u_n) (still denoted by (u_n)) and $u \in W$ such that :

$$\begin{array}{ll} u_n \rightharpoonup u & \text{weakly in } W, \\ u_n \rightharpoonup u & \text{weakly in } L^p\left(\Omega, |x - a_i|^{-p}\right) \text{ for } 1 \leq i \leq k \text{ and in } L^{p^*}(\Omega), \\ u_n \to u & \text{strongly in } L^p\left(\Omega, |x - a_i|^{s_i - p}\right) \text{ for } 1 \leq i \leq k, \\ u_n \to u & \text{strongly in } L^q(\Omega) \text{ for } 1 \leq q < p^*. \end{array}$$

and

$$\int_{\Omega} f u_n \to \int_{\Omega} f u.$$

Using a standard argument, we deduce that u is a weak solution of problem $(\mathscr{P})$.

By the Concentration-Compactness Principle [11, 12], there exist a subsequence, still denoted by (u_n), an at most countable set $\mathfrak{I}$ of different $(x_j)_{j \in \mathfrak{I}} \subset \Omega \backslash \bigcup_{j \in \mathfrak{I} \backslash \{1,...,k\}} \{a_j\}$

and sets of nonnegative numbers $\tilde{\mu}_{x_j}, \tilde{\nu}_{x_j}$ for $j \in \mathfrak{I}$; $\tilde{\mu}_{a_i}, \tilde{\gamma}_{a_i}, \tilde{\nu}_{a_i}$ for $1 \leq i \leq k$ such that :

$$\begin{aligned} |\nabla u_n|^p &\rightharpoonup d\tilde{\mu} \geq \sum_{j \in \mathfrak{I}} \tilde{\mu}_{x_j} \delta_{x_j} + \sum_{i=1}^{k} \tilde{\mu}_{a_i} \delta_{a_i} \\ \frac{|u_n|^p}{|x - a_i|^p} &\rightharpoonup d\tilde{\gamma} = \tilde{\gamma}_{a_i} \delta_{a_i} \end{aligned}$$

and

$$|u_n|^{p^*} \rightharpoonup d\tilde{\nu} = \sum_{j \in \mathfrak{I}} \tilde{\nu}_{x_j} \delta_{x_j} + \sum_{i=1}^{k} \tilde{\nu}_{a_i} \delta_{a_i}$$

where δ_x is the Dirac mass at x.

By the Sobolev-Hardy inequalities, we get

$$\tilde{\mu}_{a_i} - \mu_i \tilde{\gamma}_{a_i} \geq S_{\mu_i} \tilde{\nu}_{a_i}^{p/p^*}, \ 1 \leq i \leq k. \tag{5}$$

Claim 1 *Either* $\tilde{\nu}_{x_j} = 0$ *or* $\tilde{\nu}_{x_j} \geq S_0^{N/p}$ *for any* $j \in \Im$ *and either* $\tilde{\nu}_{a_i} = 0$ *or* $\tilde{\nu}_{a_i} \geq S_{\mu_i}^{N/p}$ *for all* $1 \leq i \leq k$.

Proof of Claim 1. In fact, let $\varepsilon > 0$ be small enough such that $a_i \notin B_{x_j}^{\varepsilon}$ for all $1 \leq j \leq k$ and $B_{x_i}^{\varepsilon} \cap B_{x_j}^{\varepsilon} = \varnothing$ for $i \neq j$, and $i, j \in \Im$.

Let ϕ_{ε}^{j} be a smooth cut-off function centred at x_j such that :

$$0 \leq \phi_{\varepsilon}^{j} \leq 1,\ \phi_{\varepsilon}^{j} = \begin{cases} 1 & \text{if } \left|x - x_j\right| < \frac{\varepsilon}{2}, \\ 0 & \text{if } \left|x - x_j\right| > \varepsilon, \end{cases} \quad \text{and } \left|\nabla \phi_{\varepsilon}^{j}\right| \leq \frac{4}{\varepsilon},$$

then

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_{\Omega} |\nabla u_n|^p \phi_{\varepsilon}^{j} = \lim_{\varepsilon\to 0} \int_{\Omega} \phi_{\varepsilon}^{j} d\tilde{\mu} \geq \tilde{\mu}_{x_j},$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_{\Omega} \frac{|u_n|^p}{|x-a_i|^p} \phi_{\varepsilon}^{j} = \lim_{\varepsilon\to 0} \int_{\Omega} \phi_{\varepsilon}^{j} d\tilde{\gamma} = 0,$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_{\Omega} |u_n|^{p^*} \phi_{\varepsilon}^{j} = \lim_{\varepsilon\to 0} \int_{\Omega} \phi_{\varepsilon}^{j} d\tilde{\nu} = \tilde{\nu}_{x_j},$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_{\Omega} |u_n|^{p-2} \nabla u_n \nabla \phi_{\varepsilon}^{j} = 0,$$

thus we have

$$0 = \lim_{\varepsilon\to 0}\lim_{n\to\infty} \left\langle J'(u_n), u_n \phi_{\varepsilon}^{j} \right\rangle \geq \tilde{\mu}_{x_j} - \tilde{\nu}_{x_j}.$$

By the Sobolev-Hardy inequalities, we get

$$S_0 \tilde{\nu}_{x_i}^{p/p^*} \leq \tilde{\mu}_{x_j},$$

hence we deduce that

$$\tilde{\nu}_{x_j} = 0 \text{ or } \tilde{\nu}_{x_j} \geq S_0^{N/p}.$$

Consider the possibility of concentration at points a_i, with $1 \leq i \leq k$.
For $\varepsilon > 0$ be small enough such that $x_j \notin B_{a_j}^{\varepsilon}$ for all $j \in \Im$ and $B_{a_i}^{\varepsilon} \cap B_{a_j}^{\varepsilon} = \varnothing$ for $i \neq j$ and $1 \leq i, j \leq k$.

Let ψ_{ε}^{i} be a smooth cut-off function centred at x_i such that

$$0 \leq \psi_{\varepsilon}^{i} \leq 1,\ \psi_{\varepsilon}^{i} = \begin{cases} 1 & \text{if } |x - x_i| < \frac{\varepsilon}{2}, \\ 0 & \text{if } |x - x_i| > \varepsilon, \end{cases} \quad \text{and } \left|\nabla \psi_{\varepsilon}^{i}\right| \leq \frac{4}{\varepsilon},$$

then

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_{\Omega} |\nabla u_n|^p \psi_{\varepsilon}^{i} = \lim_{\varepsilon\to 0} \int_{\Omega} \psi_{\varepsilon}^{i} d\tilde{\mu} \geq \tilde{\mu}_{a_i},$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_{\Omega} |u_n|^{p^*} \psi_{\varepsilon}^{i} = \lim_{\varepsilon\to 0} \int_{\Omega} \psi_{\varepsilon}^{i} d\tilde{\nu} = \tilde{\nu}_{a_i},$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_{\Omega} \frac{|u_n|^p}{|x-a_i|^p} \psi_{\varepsilon}^{i} = \lim_{\varepsilon\to 0} \int_{\Omega} \psi_{\varepsilon}^{i} d\tilde{\gamma} = \tilde{\gamma}_{a_i},$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_{\Omega} \frac{|u_n|^p}{|x-a_j|^p} \psi_{\varepsilon}^{i} = 0 \text{ for } j \neq i,$$

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \int_{\Omega} |u_n|^{p-2} \nabla u_n \nabla \psi_{\varepsilon}^{i} = 0,$$

thus we have

$$0 = \lim_{\varepsilon\to 0}\lim_{n\to\infty} \left\langle J'(u_n), u_n \psi_{\varepsilon}^{i} \right\rangle \geq \tilde{\mu}_{a_i} - \mu_i \tilde{\gamma}_{a_i} - \tilde{\nu}_{a_i}. \tag{6}$$

From (3.2) and (3.3) we deduce that

$$S_{\mu_i}\tilde{\nu}_{a_i}^{p/p^*} \leq \tilde{\nu}_{a_i}$$

and then either $\tilde{\nu}_{a_i} = 0$ or $\tilde{\nu}_{a_i} \geq S_{\mu_i}^{N/p}$ for all $1 \leq i \leq k$. ■

Consequently, from the above argument and (3.1), we conclude that :

$$\begin{aligned} c &= \lim_{n\to\infty}\left(J(u_n) - \frac{1}{2}\left\langle J'(u_n), u_n\right\rangle\right) \\ &= \frac{1}{N}\lim_{n\to\infty}\int_\Omega |u_n|^{p^*} \\ &= \frac{1}{N}\left(\sum_{j\in\Im}\tilde{\nu}_{x_j} + \sum_{i=1}^{k}\tilde{\nu}_{a_i}\right). \end{aligned}$$

If $\tilde{\nu}_{a_i} = \tilde{\nu}_{x_j} = 0$ for all $i \in \{1,...,k\}, j \in \Im$, then $c = 0$ which contradicts the assumption that $c > 0$. On the other hand, if there exists an $i \in \{1,...,k\}$ such that $\tilde{\nu}_{a_i} \neq 0$ or there exists an $j \in \Im$ with $\tilde{\nu}_{x_j} \neq 0$ then we infer that

$$c \geq \frac{1}{N}S_{\mu_l}^{N/p} = c^*.$$

Therefore J satisfies the $(PS)_c$ condition for $c < c^*$. ■

Lemma 8 *Under the condition* $(\mathscr{H}1)$, $(\mathscr{H}2)$ *and* $0 < s_i \leq s_i^*$ *there exists* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon < \varepsilon_0$ *we have*

$$\sup_{t>0} I\left(u_j + tu_{\varepsilon,l}\right) < m_j^+ + \frac{1}{N}S_{\mu_l}^{N/p}.$$

Proof. Set $g(t) := J(u_j + tu_{\varepsilon,l})$, then $g(0) = J(u_j) < m_j^+ + \frac{1}{N}S_{\mu_l}^{N/p}$ and by the continuity of g there exists $t_0 > 0$ small enough such that $g(t) < m_j^+ + \frac{1}{N}S_{\mu_l}^{N/p}$, for all $t \in (0, t_0)$. On the other hand, it is easy to see that $g(t) \to -\infty$ as $t \to +\infty$, that is, there exists $t_1 > 0$ large enough such that $g(t) < m_j^+ + \frac{1}{N}S_{\mu_l}^{N/p}$, for all $t \geq t_1$. So we only need to show that $\sup\limits_{t_0\leq t\leq t_1} g(t) < m_j^+ + \frac{1}{N}S_{\mu_l}^{N/p}$.

From the following elementary inequality satisfied for all $\alpha, \beta \in \mathbb{R}$,

$$|\alpha+\beta|^q - |\alpha|^q - |\beta|^q - q\alpha\beta\left(|\alpha|^{q-2}|\beta|^{q-2}\right) \leq C\left(\beta|\alpha|^{q-1} + \alpha|\beta|^{q-2}\right),$$

we have

$$\begin{aligned} \sup_{t_0\leq t\leq t_1} g(t) &= \sup_{t_0\leq t\leq t_1} J(u_j + tu_{\varepsilon,l}) \\ &\leq J(u_j) + \sup_{t\geq 0} J(tu_{\varepsilon,l}) + C_1\int_\Omega\left(\left|\nabla u_j\right|^{p-1}\left|\nabla u_{\varepsilon,l}\right| + \left|\nabla u_{\varepsilon,l}\right|^{p-1}\left|\nabla u_j\right|\right)dx \\ &\quad + C_2\sum_{i=1}^{k}\mu_i\int_\Omega\left(\frac{\left|u_j\right|^{p-1}\left|u_{\varepsilon,l}\right|}{|x-a_i|^p} + \frac{\left|u_{\varepsilon,l}\right|^{p-1}\left|u_j\right|}{|x-a_i|^p}\right)dx \\ &\quad + C_3\sum_{i=1}^{k}\lambda_i\int_\Omega\left(\frac{\left|u_j\right|^{p-1}\left|u_{\varepsilon,l}\right|}{|x-a_i|^{p-\alpha_i}} + \frac{\left|u_{\varepsilon,l}\right|^{p-1}\left|u_j\right|}{|x-a_i|^{p-\alpha_i}}\right)dx \\ &\quad + C_4\int_\Omega\left(\left|u_j\right|\left|u_{\varepsilon,l}\right|^{p^*-1} + \left|u_{\varepsilon,l}\right|\left|u_j\right|^{p^*-1}\right)dx. \end{aligned}$$

By $(\mathscr{H}2)$ we obtain

$$\begin{aligned}\sup_{t_0\leq t\leq t_1} J\left(tu_{\varepsilon,l}\right) &= \sup_{t>0}\left(\frac{t^p}{p}I\left(u_{\varepsilon,l}\right)-\frac{t^{p^*}}{p^*}\int_\Omega \left|u_{\varepsilon,l}\right|^{p^*}dx - t\int_\Omega fu_{\varepsilon,l}dx\right)\\ &\leq \sup_{t>0}\left(\frac{t^p}{p}\int_\Omega\left(\left|\nabla u_{\varepsilon,l}\right|^p-\sum_{i=1}^k\mu_i\frac{\left|u_{\varepsilon,l}\right|^p}{\left|x-a_i\right|^p}\right)dx-\frac{t^{p^*}}{p^*}\int_\Omega\left|u_{\varepsilon,l}\right|^{p^*}dx\right)\\ &\quad -t_1\int_\Omega fu_{\varepsilon,l}dx\\ &\leq \frac{1}{N}S_{\mu_l}^{N/p}+\mathscr{O}\left(\varepsilon^{p(B_l-\delta)}\right)-\mathscr{O}\left(\varepsilon^\theta\left|\ln(\varepsilon)\right|\right).\end{aligned}$$

From Lemma 1 and the fact that $\theta<\min\left(B_l-\delta,\delta-A_l\right)$, it follows that

$$\sup_{t_0\leq t\leq t_1} g(t) < m_j+\frac{1}{N}S_{\mu_l}^{N/p}.$$

■

Mountain pass lemma gives us a value that is below the threshold $m_j^+ + \frac{1}{N}S_{\mu_l}^{N/2}$, whose objective is to compare it with the value $m_j^- = \inf_{\mathscr{N}_j^-} I$.

Take $u_{\varepsilon,j}\in W$ such that $\left|\nabla u_{\varepsilon,j}\right|_2=1$, then by Lemma 2 we can find a unique $t_{\varepsilon,j}^+\left(u_{\varepsilon,j}\right)>0$ such that $t_{\varepsilon,j}^+u_{\varepsilon,j}\in\mathscr{N}^-$. We may use an argument similar to the previous subsection to find $t_{\varepsilon,j}^+u_{\varepsilon,j}\in\mathscr{N}_j^-$ for ε small enough and $I\left(t_{\varepsilon,j}^+u_{\varepsilon,j}\right)=\max_{t\geq t_{\varepsilon,j,\max}} I\left(tu_{\varepsilon,j}\right)$. The uniqueness of $t_{\varepsilon,j}^+$ gives that $t_{\varepsilon,j}^+(u)$ is a continuous function of u.

Set

$$U_1=\left\{v\in W \text{ such that } \|v\|<t^+\left(\frac{v}{\|v\|}\right)\right\}\cup\{0\}$$

and

$$U_2=\left\{v\in W \text{ such that } \|v\|>t^+\left(\frac{v}{\|v\|}\right)\right\}$$

we remark that $W\setminus\mathscr{N}_j^-=U_1\cup U_2$ and $\mathscr{N}_j^+\subset U_1$. In particular $u_j\in U_1$.

We claim that for t_j carefully chosen and $\varepsilon>0$ small enough $\widehat{u_j}=u_j+t_ju_{\varepsilon,j}\in U_2$ (using the same argument as [13]).

Set

$$\pounds_j=\left\{h:[0,1]\longrightarrow W \text{ continuous with } h(0)=u_j,\ h(1)=\widehat{u_j}\right\}.$$

We have :

Lemma 9 *For a suitable choice of $t_l>0$ and $\varepsilon>0$,*

$$c_j^*=\inf_{h\in\pounds_j}\max_{t\in[0,1]} I(h(t))$$

defines a critical value for I and $c_j^*\geq m_j^-$.

Proof. Clearly $h:[0,1]\longrightarrow W$ given by $h(t)=u_j+tt_ju_{\varepsilon,l}$ belongs to $\pounds_j$. Thus

$$I(h(t))<m_j^++\frac{1}{N}S_{\mu_l}^{N/p}$$

and hence

$$c_j^* < m_j^+ + \frac{1}{N} S_{\mu_l}^{N/p}.$$

Also, since the range of any $h \in \pounds_j$ intersects $\mathscr{N}_j^-$ we obtain :

$$c_j^* \geq m_j^- = \inf_{\mathscr{N}_j^-} I.$$

Lemma 7 results by applying the mountain pass lemma. ■

Proposition 10 *Suppose that f verifies the condition $(\mathscr{H}1)$ and $(\mathscr{H}2)$ then I has a minimizer $u_j \in \mathscr{N}_j^-$ such that $m_j^- = I\left(u_j\right)$. Moreover, u_j is a solution of the problem $(\mathscr{P})$.*

Proof. There exists a minimizing sequence $\left(v_{j,n}\right) \subset \mathscr{N}_j^-$ such that $I\left(v_{j,n}\right) \longrightarrow m_j^-$ and $I'\left(v_{j,n}\right) \longrightarrow 0$ in W.

By Lemma 7, we have $m_j^- < m_j^+ + \frac{1}{N} S_{\mu_l}^{N/p}$. Using Lemma 6, we deduce that $v_{j,n}$ converges strongly to u_j in W. Thus $u_j \in \mathscr{N}_j^-$ and $m_j^- = I\left(u_j\right)$.

Then $I'\left(u_j\right) = 0$, and so u_j is a solution of the problem $(\mathscr{P})$.We conclude that $(\mathscr{P})$ admits also k solutions in $\mathscr{N}^-$. ■

4. CONCLUSION

Proof of Theorem 1. By Proposition 1 and Proposition 2 we conclude that the problem$(\mathscr{P})$ admits at least $2k$ distinct solutions in W. ■

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