THE PROBLEM OF THE MIXED BOUNDARY VALUE OF THE ELASTIC MEDIUM UNDER TORSION

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ABSTRACT

The present work aims to investigate a penny-shaped crack problem in the interior of a homogeneous elastic material under an axisymmetric torsion by a circular rigid inclusion embedded in the elastic medium. With the use of the Hankel integral transformation method, the mixed boundary value problem is reduced to a system of dual integral equations. The latter is converted into a regular system of Fredholm integral equations of the second kind which is then solved by quadrature rule. Numerical results for the displacement, stress and stress intensity factor are presented graphically in some particular cases of the problem.

1. INTRODUCTION

The problems' category which examines the state of stresses and displacements in an elastic layer medium, due to a torsion of a circular inclusion in bonded contact, has been a subject of much interest in in geotechnical engineering, civil engineering and applied mechanics. It may give a better understanding of the behavior of foundations under external loads. In structure-medium interaction problems arising in foundation engineering, the foundation is usually modeled using a rigid or flexible inclusion having circular, strip, rectangular or arbitrary shape. Generally, an inclusion in contact with an elastic medium can be excited by normal translation, lateral translation, rocking rotation and torsional rotation. From a practical viewpoint, in geomechanical applications, the inclusion may represent the resinous or cementing material, which is used to transfer the anchoring loads to the geological medium.[1]. In this category of problem the penny-shaped crack can be caused by thermally induced stresses in the dilatation of the inclusion or the hydraulic fracture.

It has been shown that for foundations in which the depth of embedment exceeds the dimension of the foundation by ten times, the medium can be considered as infinite elastic space[2]. For the case of infinite embedment of the rigid disc in an infinite elastic solid (deeply embedded), Selvadurai [3, 4] investigated the asymmetric contact problems related to a rigid circular inclusion disc embedded in bonded contact with an isotropic elastic medium. Their results depend on the rotational or translational stiffnesses for the embedded rigid circular disc .

The problem on the torsion of an elastic half space was considered, at first, by Reissner and Sagoci [5]. They studied the static interaction of a rigid disc and an elastic isotropic half-space for which they obtained the solution by means of the spheroidal coordinates. Sneddon [6, 7] re-studied the classical Reissner-Sagoci problem by a different method using the Hankel transforms method for reduction the problem to a pair of dual integral equations. Ufliand [8] set up the dual integral equations for the Reissner-Sagoci problem for a circular disc on an elastic layer and reduced them to the solution of a Fredholm integral equation of the second kind. Collins [9] treated the torsional problem of an elastic half-space by supposing the displacement at any point in the half-space to be due to a distribution of wave sources over the part of the free surface in contact with the disc. The solution of the forced vibration problem of elastic layer of finite thickness when the lower face is either stress free or rigidly clamped was given by Gladwell [10] who reduced the mixed boundary value problem to a Fredholm integral equation by Noble's method [11]. Singh and Dhaliwal [12] investigated the Reissner-Sagoci problem for an elastic layer under torsion by a pair of a circular discs on opposite faces. Reissner-Sagoci problem was solved by Selvadurai [13] for a problem related to the axisymmetric rotation of a rigid circular punch which is bonded to the surface of a transversely isotropic elastic halfspace region. Pak and Saphores [14] provided an analytical formulation for the general torsional problem of a rigid disc embedded in an isotropic half-space. The quadrature numerical was used for solving the obtained Fredholm integral equation. Besides, Bacci and Bennati [15] employed the Hankel transforms method and the power series method with the truncation of the second term to consider the torsional of circular rigid disc adhered to the upper surface of an elastic layer fixed to an undeformable support.

More recently, Singh et al.[16] studied the static torsional loading of a non-homogeneous, isotropic, half-space by rotating a circular part of its boundary surface. The solution of the corresponding triple integral equations was reduced to the solution of two simultaneous integral equations. Cai and Zue [17] discussed the torsional vibration of a rigid disc bonded to a poroelastic multilayered medium. They used the Hankel transforms and transferring matrix method. Rahimian[18] et al studied the problem of torsion in a transversely isotropic half-space by a rigid circular disc. Using cylindrical co-ordinate system and applying Hankel integral transform in the radial direction, the problem may be changed to a system of dual integral equations. Yu [19] studied the forced torsional oscillations inside the multilayered solid. The elastodynamic Green's function of the center of rotation and a point load method were used to solve the problem. Pal and Mandal [20] considered the forced torsional oscillations of a transversely isotropic elastic half space under the action of an inside rigid disc. The studied problem was transformed to dual integral equations system. Which was reduced to a fredholm integral equation. A similar problem with the rocking rotation was solved later on by Ahmadi and Eskandari [21]. They used an appropiate Green's function to write the mixed boundary-value problem posed as a dual integral equation.

The torsional problem of elastic layers with a penny shaped crack was considered by some researchers. Sih and Chen [22] studied the problem of a penny-shaped crack in layered composite under a uniform torsional stress. The displacement and stress fields throughout the composite were obtained by solving a standard Fredholm integral equation of the second kind. Low [23] investigated a problem of the effects of embedded flaws in the form of an inclusion or a crack in an elastic half space subjected to torsional deformations. The corresponding Fredholm integral equations were solved numerically by quadrature approach. The same method was used by Dhawan [24] for solving the problem of a rigid disc attached to an elastic half-space with an internal crack. By using Hankel and Laplace transforms and taking numerical inversion of Laplace transform, Basu and Mandal [25] treated the torsional load on a penny-shaped crack in an elastic layer sandwiched between two elastic half-spaces.

In this paper, we investigate the problem of a penny-shaped crack in the interior of an homogeneous elastic medium under an axisymmetric torsion applied to a rigid disc glued inside. With the aid of the Hankel integral transformation method. The mixed boundary-value problem is written as a system dual of integral equations. The corresponding system of Fredholm integral equations was approached by sets of linear equations. After getting the unknown coefficients of this system we obtain numerical results and display curves according to certain pertinent parameters.

2. FORMULATION OF THE PROBLEM

We consider the axisymmetric torsion of a circular rigid inclusion of a radius b situated on plane $z = h$ in an infinite, isotropic and homogeneous elastic medium, containing a penny-shaped crack in the region $0 < r < a$, $z = 0$. The faces of the crack are supposed stress free while the rigid circular disc inclusion rotates with an angle ω about the z axis passing through their centers as shown in Fig.1.

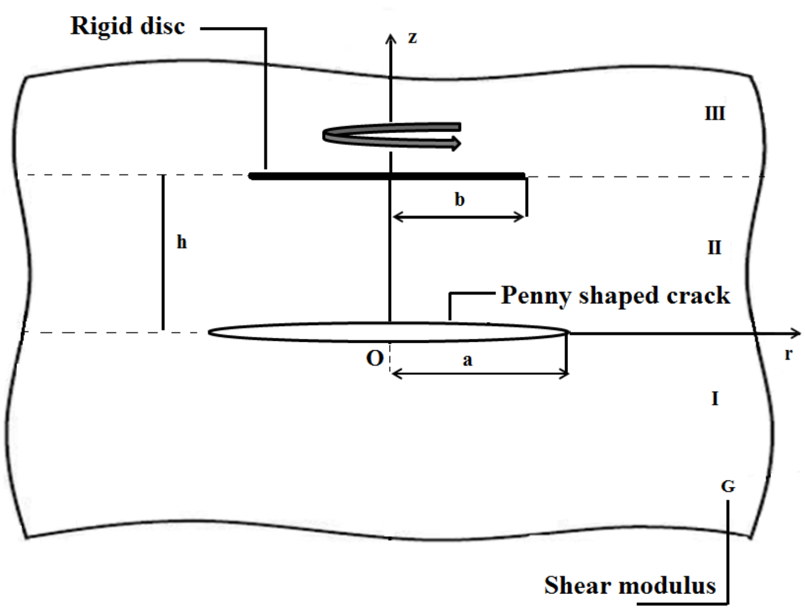


FIGURE 1 – Geometry and coordinate system

As the geometry studied is axisymmetric in the geometry and the loading (radially symmetric) where the angular displacement u_θ depend only on the r and z then the radial and axial displacement components are zero, that is $u_r = u_z = 0$.

Then the only non-zero components stresses are related to the displacement component by

$$\tau_{\theta z} = G\frac{\partial u_\theta}{\partial z}, \qquad \tau_{\theta r} = Gr\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) \tag{1}$$

where G is the shear modulus of the material.

For the static axisymmetric torsion of a homogeneous isotropic material and linear elastic behaviour, the displacement satisfies the following differential equation

$$\frac{\partial^2 u_\theta}{\partial r^2} + \frac{\partial u_\theta}{r\partial r} - \frac{u_\theta}{r^2} + \frac{\partial^2 u_\theta}{\partial z^2} = 0 \tag{2}$$

By means of the Hankel's transformation integral and its inverse given by [26]

$$F(\lambda, z) = \int_0^\infty f(r,z) r J_1(\lambda r) dr \tag{3}$$

and

$$f(r,z) = \int_0^\infty F(\lambda,z)\lambda J_1(\lambda r)d\lambda \tag{4}$$

where J_1 is the Bessel function of the first kind of order one. The general solution of Eq.(1) for the regions $I(z \leq 0), II(0 \leq z \leq h)$ and $III(z \geq h)$ as shown in Fig.1 is expressed as

$$u_\theta^{(i)}(r,z) = \int_0^\infty [A_i(\lambda)e^{-\lambda z} + B_i(\lambda)e^{\lambda z}]J_1(\lambda r)d\lambda \qquad i = 1,2,3 \tag{5}$$

where A_i and B_i are unknown functions.

3. BOUNDARY AND CONTINUITY CONDITIONS

Let us assume the contact between the rigid circular inclusion and the elastic layer is perfectly bonded all along their common interface. We consider the regularity conditions at infinity, the boundary and continuity conditions at $z = h$, as shown in the following

At infinity, the regularity conditions are given by

$$\lim_{|z|\to\infty} u_\theta(r,z) = 0, \qquad \lim_{|z|\to\infty} \tau_{\theta z}(r,z) = 0 \tag{6}$$

The boundary conditions of the problem are

$$\tau_{\theta z}^{(2)}(r,0^+) = \tau_{\theta z}^{(1)}(r,0^-) = 0 \qquad r < a \tag{7a}$$

$$u_\theta^{(3)}(r,h) = u_\theta^{(2)}(r,h) = \omega r \qquad r \leq b \tag{7b}$$

The continuity conditions of the problem in the planes $z = 0$ and $z = h$ can be written as

$$u_\theta^{(2)}(r,0^+) - u_\theta^{(1)}(r,0^-) = 0 \qquad r \geq a \tag{8a}$$

$$\tau_{\theta z}^{(2)}(r,0^+) - \tau_{\theta z}^{(1)}(r,0^-) = 0 \qquad r \geq a \tag{8b}$$

$$u_\theta^{(3)}(r,h^+) - u_\theta^{(2)}(r,h^-) = 0 \qquad r > b \tag{8c}$$

$$\tau_{\theta z}^{(3)}(r,h^+) - \tau_{\theta z}^{(2)}(r,h^-) = 0 \qquad r > b \tag{8d}$$

By utilizing the condition Eq.(6), the expressions of displacements and stresses in the three regions take the following forms

$$u_\theta^{(1)}(r,z) = \int_0^\infty B_1(\lambda)e^{\lambda z}J_1(\lambda r)d\lambda \tag{9a}$$

$$\tau_{\theta z}^{(1)}(r,z) = G\int_0^\infty \lambda B_1(\lambda)e^{\lambda z}J_1(\lambda r)d\lambda \tag{9b}$$

$$u_\theta^{(2)}(r,z) = \int_0^\infty \left[A_2(\lambda)e^{-\lambda z} + B_2(\lambda)e^{\lambda z}\right]J_1(\lambda r)d\lambda \tag{9c}$$

$$\tau_{\theta z}^{(2)}(r,z) = G\int_0^\infty \lambda\left[-A_2(\lambda)e^{-\lambda z} + B_2(\lambda)e^{\lambda z}\right]J_1(\lambda r)d\lambda \tag{9d}$$

$$u_\theta^{(3)}(r,z) = \int_0^\infty A_3(\lambda)e^{-\lambda z}J_1(\lambda r)d\lambda \tag{9e}$$

$$\tau_{\theta z}^{(3)}(r,z) = -G\int_0^\infty \lambda A_3(\lambda)e^{-\lambda z}J_1(\lambda r)d\lambda \tag{9f}$$

The unknown functions $B_1(\lambda)$, $A_2(\lambda)$, $B_2(\lambda)$ and $A_3(\lambda)$ can be determined from the boundary and continuity conditions.

The boundary and continuity conditions Eqs. (7a), (8b), (7b) and (8c) show that

$$\tau^{(2)}_{\theta z}(r,0^+) - \tau^{(1)}_{\theta z}(r,0^-) = 0 \qquad r \geq 0 \tag{10a}$$

$$u^{(3)}_{\theta}(r,h^+) - u^{(2)}_{\theta}(r,h^-) = 0 \qquad r \geq 0 \tag{10b}$$

The continuity conditions Eqs.(8b) and (8c) lead to

$$B_1(\lambda) = B_2(\lambda) - A_2(\lambda) \tag{11a}$$

$$A_3(\lambda) = B_2(\lambda)e^{2\lambda h} + A_2(\lambda) \tag{11b}$$

From the mixed boundary conditions Eqs.(7a), (8a), (7b) and (8d), we find the system of dual integral equations for obtained the unknown functions A_2 and B_2

$$\int_0^\infty \lambda[B_2(\lambda) - A_2(\lambda)]J_1(\lambda r)d\lambda = 0,\ r < a \tag{12a}$$

$$\int_0^\infty A_2(\lambda)J_1(\lambda r)d\lambda = 0,\ r \geq a \tag{12b}$$

$$\int_0^\infty [A_2(\lambda)e^{-\lambda h} + B_2(\lambda)e^{\lambda h}]J_1(\lambda r)d\lambda = \omega r,\ r \leq b \tag{12c}$$

$$\int_0^\infty \lambda B_2(\lambda)e^{\lambda h}J_1(\lambda r)d\lambda = 0,\ r > b \tag{12d}$$

3.1. Limiting Cases

By taking the limit $a \to \infty$, the problem is simplified to the torsional rotation of a rigid cirular inclusion in a homogeneous elastic half-space, the dual integral equations become :

$$\int_0^\infty [A_2(\lambda)e^{-\lambda h} + B_2(\lambda)e^{\lambda h}]J_1(\lambda r)d\lambda = \omega r,\ r \leq b \tag{13a}$$

$$\int_0^\infty \lambda B_2(\lambda)e^{\lambda h}J_1(\lambda r)d\lambda = 0,\ r > b \tag{13b}$$

This pair of dual integral equations has the same meaning as $(18a)$ and $(18b)$ in Pak's paper [14].

Let's take the limit $a \to 0$, one can obtain the closed-form solution pertinent to the torsional rotation of a rigid disc embedded in a homogeneous elastic full-space. Due to the symmetry of the full-space case with respect to the plane of the disc, it can be deduced that $\tau_\theta z$ is zero for $r > a$ at the disc plane. This situation corresponds exactly to the torsion of a homogeneous elastic half-space by a circular rigid disc $(0 < r < a, z = 0)$ bonded to the surface. This is adapted to the problem concerning isotropic half-space considered by Reissner and Sagoci [5].

4. REDUCTION OF THE PROBLEM TO A SYSTEM OF FREDHOLM INTEGRAL EQUATIONS

The system of dual equations can be reduced to a system of Fredholm integral equations of second kind by introducing the auxiliary functions $\phi(t)$ and $\psi(t)$ such that

$$A_2(\lambda) = \sqrt{\lambda}\int_0^a \sqrt{t}\phi(t)J_{\frac{3}{2}}(\lambda t)dt \tag{14a}$$

$$B_2(\lambda) = e^{-\lambda h}\sqrt{\lambda}\int_0^b \sqrt{t}\psi(t)J_{\frac{1}{2}}(\lambda t)dt \tag{14b}$$

With this choice of the new unknown functions, we find that the homogeneous equations Eq.(12b) and Eq.(12d) are identically satisfied while equations Eq.(12a) and Eq.(12c) lead to the Fredholm's integral equations.

By inserting $A_2(\lambda)$ and $B_2(\lambda)$ in the equations Eq.(12a) and Eq.(12c), we get

$$\int_0^a \sqrt{t}\phi(t)dt \int_0^\infty \lambda^{\frac{3}{2}} J_{\frac{3}{2}}(\lambda t)J_1(\lambda r)d\lambda - \int_0^b \sqrt{t}\psi(t)dt \int_0^\infty \lambda^{\frac{3}{2}} e^{-\lambda h} J_{\frac{1}{2}}(\lambda t)J_1(\lambda r)d\lambda = 0,\ r<a \quad (15)$$

$$\int_0^a \sqrt{t}\phi(t)dt \int_0^\infty \sqrt{\lambda} e^{-\lambda h} J_{\frac{3}{2}}(\lambda t)J_1(\lambda r)d\lambda + \int_0^b \sqrt{t}\psi(t)dt \int_0^\infty \sqrt{\lambda} J_{\frac{1}{2}}(\lambda t)J_1(\lambda r)d\lambda = \omega r,\ r<b \quad (16)$$

To find the first Fredholm integral equation, we use $\lambda J_1(\lambda r) = \frac{1}{r^2}\frac{d}{dr}[r^2 J_2(\lambda r)]$.

Taking into account the integral formula

$$\int_0^\infty \sqrt{\lambda} J_{\frac{3}{2}}(\lambda t)J_2(\lambda r)d\lambda = \begin{cases} \sqrt{\frac{2}{\pi}} \frac{t^{\frac{3}{2}}}{r^2\sqrt{r^2-t^2}} & t<r \\ 0 & t>r \end{cases}$$

we obtain Abel equation corresponding to equation Eq.(15)

$$\sqrt{\frac{2}{\pi}} \int_0^r \frac{t^2\phi(t)}{\sqrt{r^2-t^2}}dt - r^2 \int_0^b \sqrt{t}\psi(t)dt \int_0^\infty \sqrt{\lambda} e^{-\lambda h} J_{\frac{1}{2}}(\lambda t)J_2(\lambda r)d\lambda = 0,\ r<a \quad (17)$$

By applying the Abel's transform formula

$$\int_0^r \frac{f(t)}{\sqrt{r^2-t^2}}dt = g(r) \quad \text{then} \quad f(t) = \frac{2}{\pi}\frac{d}{dt}\int_0^t \frac{rg(r)}{\sqrt{t^2-r^2}}dr$$

we then find from Eq.(**??**) that

$$t^2\phi(t) = \sqrt{\frac{2}{\pi}}\frac{d}{dt}\int_0^t \frac{r^3}{\sqrt{t^2-r^2}} \left[\int_0^b \sqrt{\delta}\psi(\delta)d\delta \int_0^\infty \sqrt{\lambda} e^{-\lambda h} J_{\frac{1}{2}}(\lambda\delta)J_2(\lambda r)d\lambda dr,\ r<a \right. \quad (18)$$

For the right hand side of the above equation, the integral is further simplified by using the following relationship

$$\sqrt{\frac{2}{\pi}}\frac{d}{dt}\int_0^t \frac{r^3}{\sqrt{t^2-r^2}} J_2(\lambda r)dr = \sqrt{\lambda} t^{\frac{5}{2}} J_{\frac{3}{2}}(\lambda t)$$

we obtain the first Fredholm integral equation of second kind

$$\phi(t) + \sqrt{t}\int_0^b \sqrt{\delta}\psi(\delta)K(t,\delta)d\delta = 0,\ r<a \quad (19)$$

where

$$K(t,\delta)=-\int_0^\infty \lambda e^{-\lambda h}J_{\frac{3}{2}}(\lambda t)J_{\frac{1}{2}}(\lambda\delta)d\lambda$$

Similarly, Eq.(**??**) can be reduced to the second Fredholm integral equation as follows
By using the formula

$$\int_0^\infty \sqrt{\lambda}J_{\frac{1}{2}}(\lambda t)J_1(\lambda r)d\lambda=\begin{cases}\sqrt{\frac{2t}{\pi}}\frac{1}{r\sqrt{r^2-t^2}} & t<r\\ 0 & t>r\end{cases}$$

we obtain the following Abel equation

$$\frac{1}{r}\sqrt{\frac{2}{\pi}}\int_0^r \frac{t\psi(t)}{\sqrt{r^2-t^2}}dt+\int_0^a \sqrt{t}\phi(t)dt \int_0^\infty \sqrt{\lambda}e^{-\lambda h}J_{\frac{3}{2}}(\lambda t)J_1(\lambda r)d\lambda=\omega r,\ r<b \quad (20)$$

By applying the Abel's transform formula to the last equation, we obtain

$$t\psi(t)=\sqrt{\frac{2}{\pi}}\frac{d}{dt}\int_0^t \frac{r^2}{\sqrt{t^2-r^2}}\left[\omega r-\int_0^a \sqrt{\delta}\phi(\delta)d\delta \int_0^\infty \sqrt{\lambda}e^{-\lambda h}J_{\frac{3}{2}}(\lambda\delta)J_1(\lambda r)d\lambda dr,\ r<b \right. \quad (21)$$

Using the following relationship

$$\frac{d}{dt}\int_0^t \frac{r^3}{\sqrt{t^2-r^2}}dr=2t^2$$

$$\sqrt{\frac{2}{\pi}}\frac{d}{dt}\int_0^t \frac{r^2J_1(\lambda r)}{\sqrt{t^2-r^2}}dr=t\sqrt{\lambda t}J_{\frac{1}{2}}(\lambda t)$$

we finally get the second Fredholm integral equation of second kind

$$\psi(t)+\sqrt{t}\int_0^a \sqrt{\delta}\phi(\delta)L(t,\delta)d\delta=\frac{4\omega}{\sqrt{2\pi}}t,\ t<b \quad (22)$$

with the kernel

$$L(t,\delta)=\int_0^\infty \lambda e^{-\lambda h}J_{\frac{1}{2}}(\lambda t)J_{\frac{3}{2}}(\lambda\delta)d\lambda$$

The system given by Eq.(
By putting

$$\begin{cases}\delta=a\eta, & 0<\delta<a; \quad t=a\xi \quad 0<t<a\\ \delta=b\eta, & 0<\delta<b; \quad t=b\xi \quad 0<t<b\end{cases} \quad (23)$$

Next, we multiply the above two equations of the system, respectively by $\frac{\sqrt{2\pi}}{4a\omega}\phi(au)$ and $\frac{\sqrt{2\pi}}{4b\omega}\psi(bu)$ and using the following substitutions

$$\begin{cases}\Phi(u)=\frac{\sqrt{2\pi}}{4a\omega}\phi(au) \quad \Psi(u)=\frac{\sqrt{2\pi}}{4b\omega}\psi(bu)\\ c=\frac{b}{a} \quad \lambda=\frac{x}{a} \quad H=\frac{h}{a} \quad \rho=\frac{r}{a} \quad \zeta=\frac{z}{a}\end{cases} \quad (24)$$

we obtain

$$\Phi(\xi)+c^2\sqrt{c}\sqrt{\xi}\int_0^1\sqrt{\eta}\Psi(s)K(\xi,\eta)d\eta=0, \qquad \xi<1 \tag{25}$$

$$\Psi(\xi)+\frac{1}{\sqrt{c}}\sqrt{\xi}\int_0^1\sqrt{\eta}\Phi(\eta)L(\xi,\eta)d\eta=\xi, \qquad \xi<1 \tag{26}$$

where

$$K(\xi,\eta)=-\int_0^\infty xe^{-xH}J_{\frac{3}{2}}(x\xi)J_{\frac{1}{2}}(xc\eta)dx$$
$$=-\frac{2}{\pi}\frac{1}{\sqrt{c\xi\eta}}\int_0^\infty e^{-xH}sin(xc\eta)[\frac{sin(x\xi)}{x\xi}-cos(x\xi)]dx$$

$$L(\xi,\eta)=\int_0^\infty xe^{-xH}J_{\frac{1}{2}}(xc\xi)J_{\frac{3}{2}}(x\eta)dx$$
$$=\frac{2}{\pi}\frac{1}{\sqrt{c\xi\eta}}\int_0^\infty e^{-xH}sin(xc\xi)[\frac{sin(x\eta)}{x\eta}-cos(x\eta)]dx$$

The indefinite integrals K and L can be evaluated in closed form given in (3 :947 :1-2), (3 :948 :2) and (3 :893 :1-2) from [27], we obtain

$$K(\xi,\eta)=-\frac{1}{\pi\sqrt{\xi c\eta}}[\frac{1}{2\xi}\log\frac{H^2+(c\eta+\xi)^2}{H^2+(c\eta-\xi)^2}-$$
$$(\frac{c\eta+\xi}{H^2+(c\eta+\xi)^2}+\frac{c\eta-\xi}{H^2+(c\eta+\xi)^2})] \tag{27a}$$

$$L(\xi,\eta)=\frac{1}{\pi\sqrt{\eta c\xi}}[\frac{1}{2\eta}\log\frac{H^2+(c\xi+\eta)^2}{H^2+(c\xi-\eta)^2}-$$
$$(\frac{c\xi+\eta}{H^2+(c\xi+\eta)^2}+\frac{c\xi-\eta}{H^2+(c\xi+\eta)^2})] \tag{27b}$$

5. NUMERICAL RESULTS AND DISCUSSION

As the kernels K and L are continuous on the interval $[0,1]$, the system of Fredholm integral equations can be solved by direct or iterative techniques[28]. The midpoint quadrature [29] is used to find the numerical solution for the system given by Eq.(25) and Eq.(26). Dividing the interval $[0,1]$ into N equal sub-intervals, so the midpoints are $u=u_m=\frac{2m-1}{2N}$ $\quad s=u_n=\frac{2n-1}{2N}$ $\quad m,n=1,2...,N$
and introducing the following notations

$$\Phi(u_m)=\Phi_m \qquad \Psi(u_m)=\Psi_m \tag{28a}$$

$$K(u_m,u_n)=K_{mn} \qquad L(u_m,u_n)=L_{mn} \tag{28b}$$

we obtain the following systems of finite algebraic equations

$$\Phi_m+\frac{c^2\sqrt{c}}{N}\sqrt{u_m}\sum_{n=1}^{N}\sqrt{u_n}\Psi_nK_{mn}=0, m=1,2,...,N \tag{29}$$

$$\Psi_m+\frac{1}{N\sqrt{c}}\sqrt{u_m}\sum_{n=1}^{N}\sqrt{u_n}\Phi_nL_{mn}=u_m, m=1,2,...,N \tag{30}$$

After solving the above system, the unknown coeffcients can be obtained then we ge the numerical approximation of the unknown functions B_1, A_2, B_2 and A_3 given by Eq.(11a), Eq.(14a), Eq.(14b) and Eq.(11b)

$$B_1(x) = \frac{4a^2\omega}{N\sqrt{2\pi}}\sqrt{x}\sum_{m=1}^{N}\sqrt{u_m}[e^{-xH}c^2\sqrt{c}$$
$$\Psi_m J_{\frac{1}{2}}(xcu_m) - \Phi_m J_{\frac{3}{2}}(xu_m)] \quad (31a)$$

$$A_2(x) = \frac{4a^2\omega}{N\sqrt{2\pi}}\sqrt{x}\sum_{m=1}^{N}\sqrt{u_m}\Phi_m J_{\frac{3}{2}}(xu_m) \quad (31b)$$

$$B_2(x) = e^{-xH}\frac{4b^2\sqrt{c}\omega}{N\sqrt{2\pi}}\sqrt{x}\sum_{m=1}^{N}\sqrt{u_m}\Psi_m J_{\frac{1}{2}}(xcu_m) \quad (31c)$$

$$A_3(x) = \frac{4a^2\omega}{N\sqrt{2\pi}}\sqrt{x}\sum_{m=1}^{N}\sqrt{u_m}[e^{xH}c^2\sqrt{c}$$
$$\Psi_m J_{\frac{1}{2}}(xcu_m) + \Phi_m J_{\frac{3}{2}}(xu_m)] \quad (31d)$$

5.1. Stress intensity factor

The stress intensity factor at the edge of the crack and at the rim of the disc are defined respectively by

$$K^a_{\text{III}} = \lim_{r\to a^+}\sqrt{2\pi(r-a)}\tau^{(2)}_{\theta z}(r,z)|_{z=0} \quad (32)$$

$$K^b_{\text{III}} = \lim_{r\to b^-}\sqrt{2\pi(b-r)}\tau^{(2)}_{\theta z}(r,z)|_{z=h} \quad (33)$$

On the plane $z = 0$ for $r \geq a$, the expression of stress is given by

$$\tau^{(2)}_{\theta z}(r,0) = G\int_0^\infty[-\lambda^{\frac{3}{2}}\int_0^a\sqrt{t}\phi(t)J_{\frac{3}{2}}(\lambda t)dt +$$
$$e^{-\lambda h}\lambda^{\frac{3}{2}}\int_0^b\sqrt{t}\psi(t)J_{\frac{1}{2}}(\lambda t)dt]J_1(\lambda r)d\lambda \quad (34)$$

On the plane $z = h$, the expression of stress is given by

$$\tau^{(2)}_{\theta z}(r,h) = G\int_0^\infty[-e^{-\lambda h}\lambda^{\frac{3}{2}}\int_0^a\sqrt{t}\phi(t)J_{\frac{3}{2}}(\lambda t)dt +$$
$$\lambda^{\frac{3}{2}}\int_0^b\sqrt{t}\psi(t)J_{\frac{1}{2}}(\lambda t)dt]J_1(\lambda r)d\lambda \quad (35)$$

The second and the first part of the integrals 34 and 35 respectively converge quickly, their limits as $r \to a$ and $r \to b$ automatically vanishe, although, the limits of the other two integrals analyzed asymptotically as follows

Using the relation

$$J_1(\lambda r) = -\frac{1}{\lambda}\frac{d}{dr}J_0(\lambda r)$$

we obtain

$$\tau_{\theta z}^{(2)}(r,0) = G\int_0^a \sqrt{t}\phi(t)dt\int_0^\infty F(\lambda,r)d\lambda + G\int_0^b \sqrt{t}\psi(t)dt\int_0^\infty e^{-\lambda h}\lambda^{\frac{3}{2}}J_{\frac{1}{2}}(\lambda t)J_1(\lambda r)d\lambda \quad (36)$$

and

$$\tau_{\theta z}^{(2)}(r,h) = -G\int_0^a \sqrt{t}\phi(t)dt\int_0^\infty e^{-\lambda h}\lambda^{\frac{3}{2}}J_{\frac{3}{2}}(\lambda t)J_1(\lambda r)d\lambda - G\int_0^b \sqrt{t}\psi(t)dt\int_0^\infty G(\lambda)d\lambda \quad (37)$$

where

$$F(\lambda,r) = \lambda^{\frac{1}{2}}J_{\frac{3}{2}}(\lambda t)J_0(\lambda r)$$

$$G(\lambda,r) = \lambda^{\frac{1}{2}}J_{\frac{1}{2}}(\lambda t)J_0(\lambda r)$$

We use the following asymptotic behavior of the Bessel function of the first kind, for large values of λ

$$J_\nu(\lambda) \simeq \sqrt{\frac{2}{\lambda\pi}}\cos(\lambda - \frac{\pi}{2}\nu - \frac{\pi}{4})$$

then we get,

$$J_{3/2}(\lambda t) \simeq \sqrt{\frac{2}{\lambda t\pi}}\cos(\lambda t - \pi) = -\sqrt{\frac{2}{\lambda t\pi}}\cos(\lambda t)$$

$$J_{1/2}(\lambda t) \simeq \sqrt{\frac{2}{\lambda t\pi}}\cos(\lambda t - \frac{\pi}{2}) = \sqrt{\frac{2}{\lambda t\pi}}\sin(\lambda t)$$

To calculate the limit of the integral

$$\lim_{r\to a^+} 2\pi\sqrt{(r-a)}\int_0^\infty F(\lambda,r)d\lambda \quad (38)$$

$$\lim_{r\to b^-} 2\pi\sqrt{(b-r)}\int_0^\infty G(\lambda,r)d\lambda \quad (39)$$

we use the asymptotic functions $F^{'}$ and $G^{'}$, we obtain

$$\int_0^\infty F(\lambda,r) = \int_0^\infty \left[F(\lambda,r) - F^{'}(\lambda,r)\right]d\lambda + \int_0^\infty F^{'}(\lambda,r)d\lambda \quad (40)$$

$$\int_0^\infty G(\lambda,r) = \int_0^\infty \left[G(\lambda,r) - G^{'}(\lambda,r)\right]d\lambda + \int_0^\infty G^{'}(\lambda,r)d\lambda \quad (41)$$

From the uniform convergence of the integral we find that

$$\lim_{r\to a^+} 2\pi\sqrt{(r-a)}\int_0^\infty \left[F(\lambda,r) - F^{'}(\lambda,r)\right]d\lambda = 0 \quad (42)$$

$$\lim_{r\to b^-} 2\pi\sqrt{(b-r)}\int_0^\infty \left[G(\lambda,r) - G^{'}(\lambda,r)\right]d\lambda = 0 \quad (43)$$

Next, we use the following integral formulas to replace the first infinite integral respectively in the right part of Eq.(36) and Eq.(37)

$$\int_0^\infty cos(\lambda t)J_0(\lambda r)d\lambda = \begin{cases} \frac{1}{\sqrt{r^2-t^2}} & r > t \\ 0 & r < t \end{cases}$$

$$\int_0^\infty sin(\lambda t)J_0(\lambda r)d\lambda = \begin{cases} 0 & r > t \\ \frac{1}{\sqrt{t^2-r^2}} & r < t \end{cases}$$

we obtain

$$\tau_{\theta z}^{(2)}(r,0) = -\sqrt{\frac{2}{\pi}}G\frac{d}{dr}\int_0^a \frac{\phi(t)}{\sqrt{r^2-t^2}}dt + R_1(r) \quad (44)$$

$$\tau_{\theta z}^{(2)}(r,h) = -\sqrt{\frac{2}{\pi}}G\frac{d}{dr}\int_0^b \frac{\psi(t)}{\sqrt{t^2-r^2}}dt + R_2(r) \quad (45)$$

where

$$R_1(r) = G\int_0^b \sqrt{t}\psi(t)dt\int_0^\infty e^{-\lambda h}\lambda^{\frac{3}{2}}J_{\frac{1}{2}}(\lambda t)J_1(\lambda r)d\lambda$$

$$R_2(r) = -G\int_0^a \sqrt{t}\phi(t)dt\int_0^\infty e^{-\lambda h}\lambda^{\frac{3}{2}}J_{\frac{3}{2}}(\lambda t)J_1(\lambda r)d\lambda$$

Now integrating by parts, we get

$$\tau_{\theta z}^{(2)}(r,0) = G\sqrt{\frac{2}{\pi}}[\frac{a\phi(a)}{r\sqrt{r^2-a^2}} - \int_0^a \frac{t\phi'(t)}{r\sqrt{r^2-t^2}}dt] + R_1(r) \quad (46)$$

We note that the infinite integrals in the preceding expressions are convergent throughout the medium except at the singular points $r \to a^+$ which occupy the crack boundary.

$$\tau_{\theta z}^{(2)}(r,h) = G\sqrt{\frac{2}{\pi}}[\frac{b\psi(b)}{r\sqrt{b^2-r^2}} - \int_r^b \frac{1}{r}\frac{t\psi'(t)}{\sqrt{t^2-r^2}}dt] + R_2(r) \quad (47)$$

In this case, the equation 47 shows that $\tau_{\theta z}^{(2)}(r,h)$ is $0(r)$ as $r \to 0$ and the integral is bonded as $r \to b^-$. As a result we obtain a square root singularity at $r = b$ and the constant $\psi(b)$ is the measure of the strength of singularity at the vicinity of the rigid inclusion.

The stress intensity factor at the edge of the rigid inclusion may be calculated as

$$K_{\rm III}^b = \lim_{r\to b^-}\sqrt{2\pi(b-r)}\frac{G\sqrt{2}}{\sqrt{\pi}}\frac{b\psi(b)}{r\sqrt{b^2-r^2}} \quad (48)$$

By using the following transformations
$\phi(a) = \dfrac{4a\omega}{\sqrt{2\pi}}\Phi_N, \qquad \psi(a) = \dfrac{4b\omega}{\sqrt{2\pi}}\Psi_N$
we obtain

$$K_{\mathrm{III}}^{a} = \frac{4G\omega\sqrt{a}}{\sqrt{\pi}}\Phi_N \tag{49}$$

$$K_{\mathrm{III}}^{b} = \frac{4G\omega\sqrt{b}}{\sqrt{\pi}}\Psi_N \tag{50}$$

Fig.2 shows the results of the effect of the normalized crack size a/b on the stress intensity factor K_{III}^{a} for different disc locations $H = 1, 0.75, 0.5$ and 0.25. It is observed that the values of the stress intensity factor versus a/b increase, attain its maximum values and then decrease to zero.

The effect of the distance between the crack and the rigid inclusion H on the stress intensity factor is also shown in Fig.2. The increase of the height H induces the decrease of stress intensity factor for all values of a/b.

Fig.3 illustrates the variation of the normalized stress intensity factor K_{III}^{b} at the edge of the rigid inclusion defined by Eq.(50) versus a/b for $H = 1, 0.75, 0.5$ and 0.25. It can be seen that the stress intensity factor starts with the value $4/\sqrt{\pi}$ which is The stress intensity factor of at the vinicity of the rigid inclusion $(a0)$ for a rigid disc alone in the infinite medium(not cracked). . Furthermore, it first increases and then decreases to a minimum value and finally increases to $4/\sqrt{\pi}$. In addition, the interaction between the inclusion and the crack is small for smaller values of a/b and the values of the stress intensity factor are greater when the crack is closer to the disc.

5.2. Displacement and stress fields

By substituting the Eqs.(31a)-(31d) into the expressions of the displacements and the stresses Eqs.(9a)-(9f), we get the numerical results of displacements and stresses for the three regions.

The results for the variation of the normalized displacements $u_{\theta}^{(i)}(\rho,\zeta)/\omega a$ and the normalized stresses $\tau^{(i)}(\rho,\zeta)/\,G\omega a$ versus the normalized radius ρ are shown graphically in Fig.4 to Fig.9 for the different values of the dimensionless axial distances $\zeta = z/a$. For each region, five different axial distances are selected as $I(\zeta = -H\,; -3H/4\,; -H/2\,; -H/4\,; 0)$, $II(\zeta = 0\,; H/4\,; H/2\,; 3H/4\,;H)$, $III(\zeta = H\,; 5H/4\,; 3H/2\ 7H/4\,;2H)$, with the particular values of the height $H = 1$ and the dimensionless disc sizes $c = 1$ and $c = 0.5$.

The variation of the normalized displacements are shown in Fig.4 to Fig.6. We notice that the displacements in the three regions increase at first, reach maximum values at $\rho = c$ in region 2 and 3 and then decrease out of the disc band with increasing ρ.

The distribution of the shear stresses in the elastic medium are also discussed and shown in Fig.7 to Fig.9. It is concluded that the magnitude of the stress in the first region is the lowest for the three others and that the stress are initially rises, attains its maximum values and with the increase in the value of ρ the stress go on decreasing.

6. CONCLUSION

In this study, the axisymmetric torsion problem of a circular rigid inclusion embedded in the interior of an homogeneous elastic material is analytically addressed. The medium is weakened by a penny-shaped crack located parallel to the plane of the inclusion. Using the Hankel integral transformation method, the doubly mixed boundary value problem is reduced to a system of dual integral equations, which are transformed, to a Fredholm integral equations system of the second kind. The presented graphs show the variation of the displacements, the stresses in the three regions and the stress intensity factor at the edge of the crack and at the rim of the inclusion

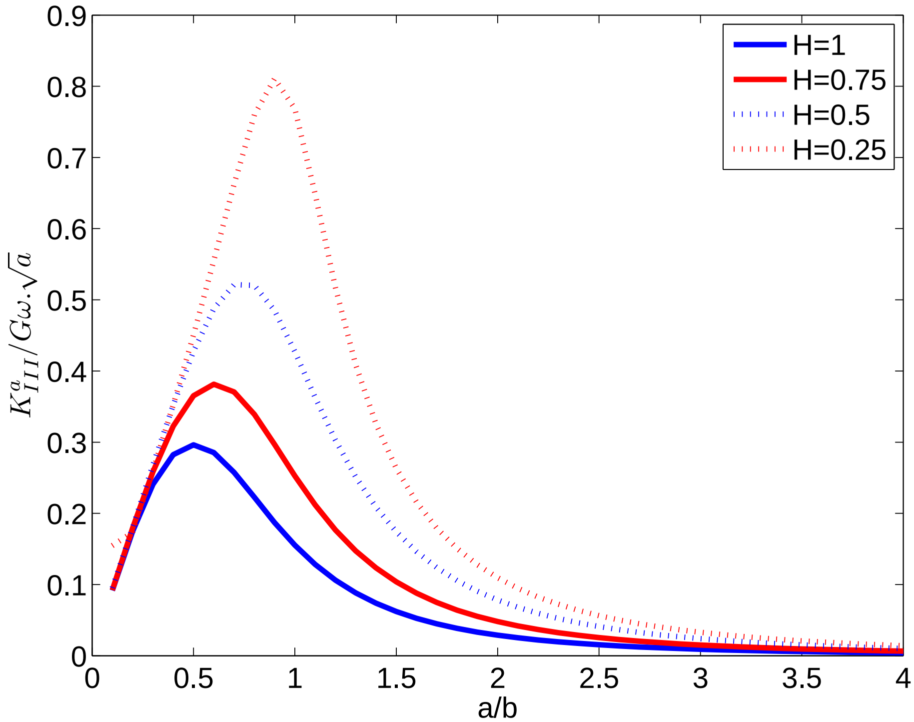


FIGURE 2 – Variation of the normalized stress intensity factor at the edge of the crack K^a_{III} with a/b

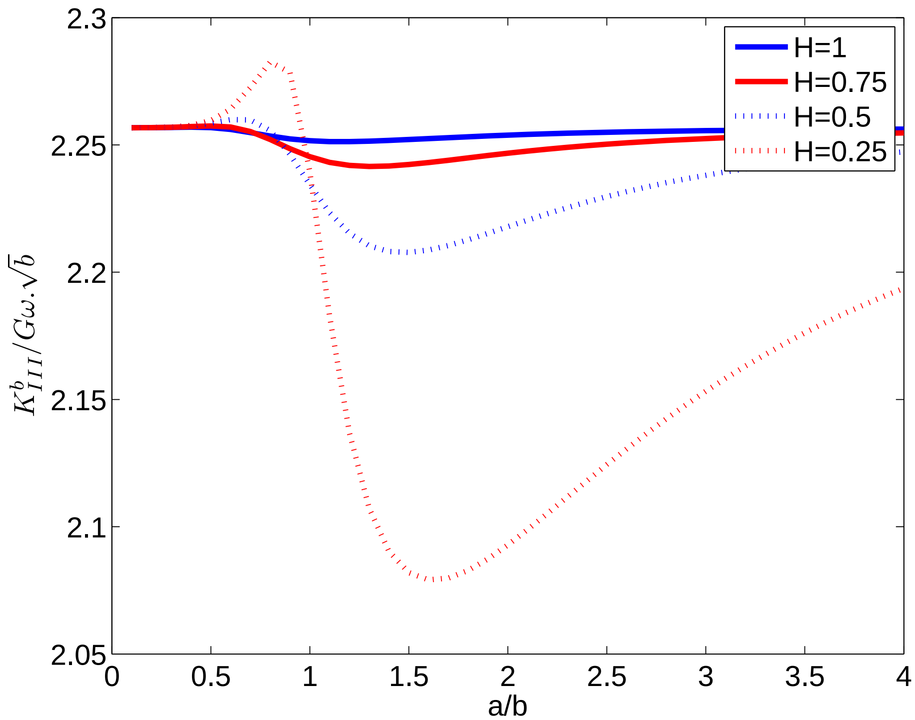


FIGURE 3 – Variation of the normalized stress intensity factor at the edge of the rigid inclusion K^b_{III} with a/b

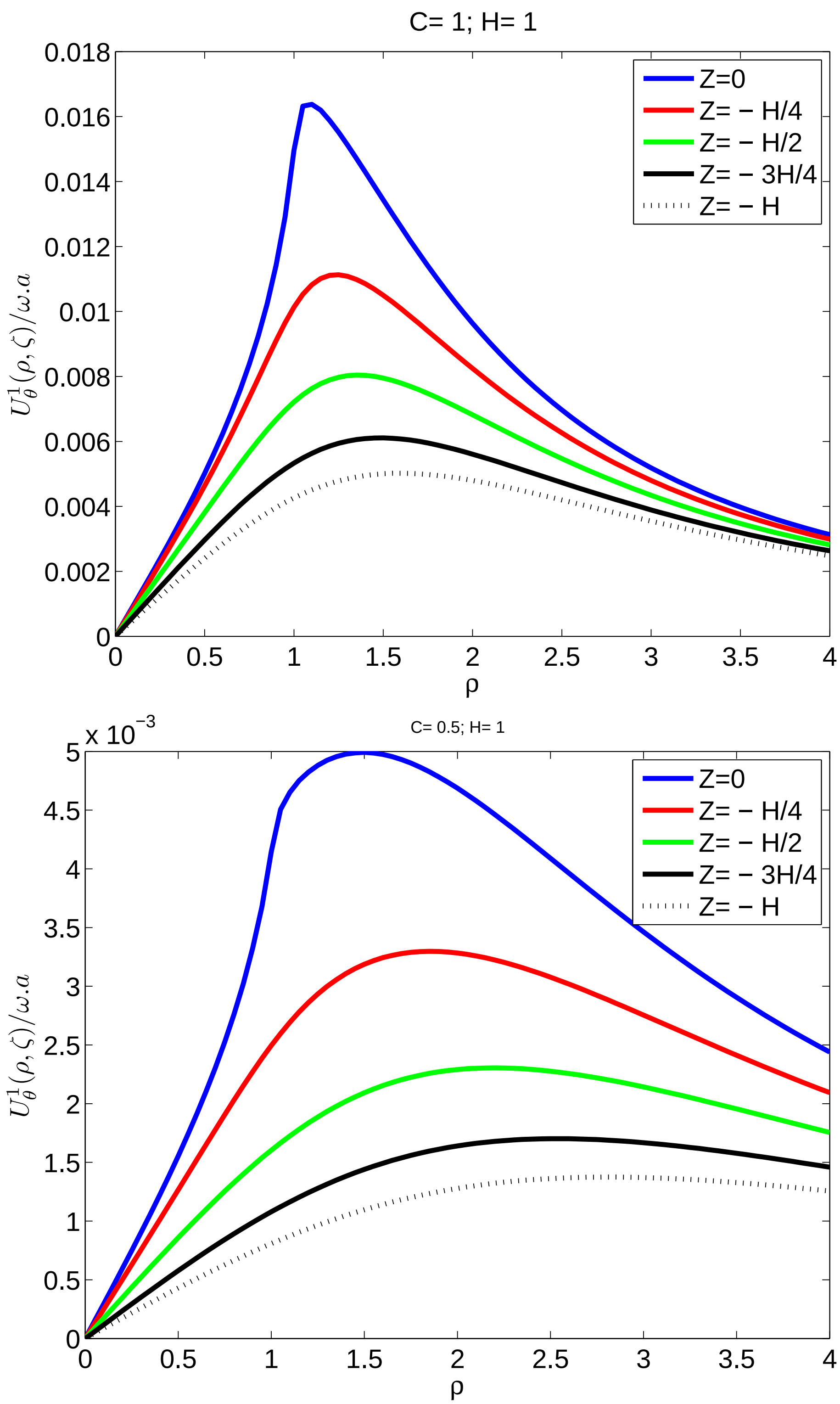


FIGURE 4 – Tangential displacement u_θ^1 versus ρ for various ζ, $z \leq 0$

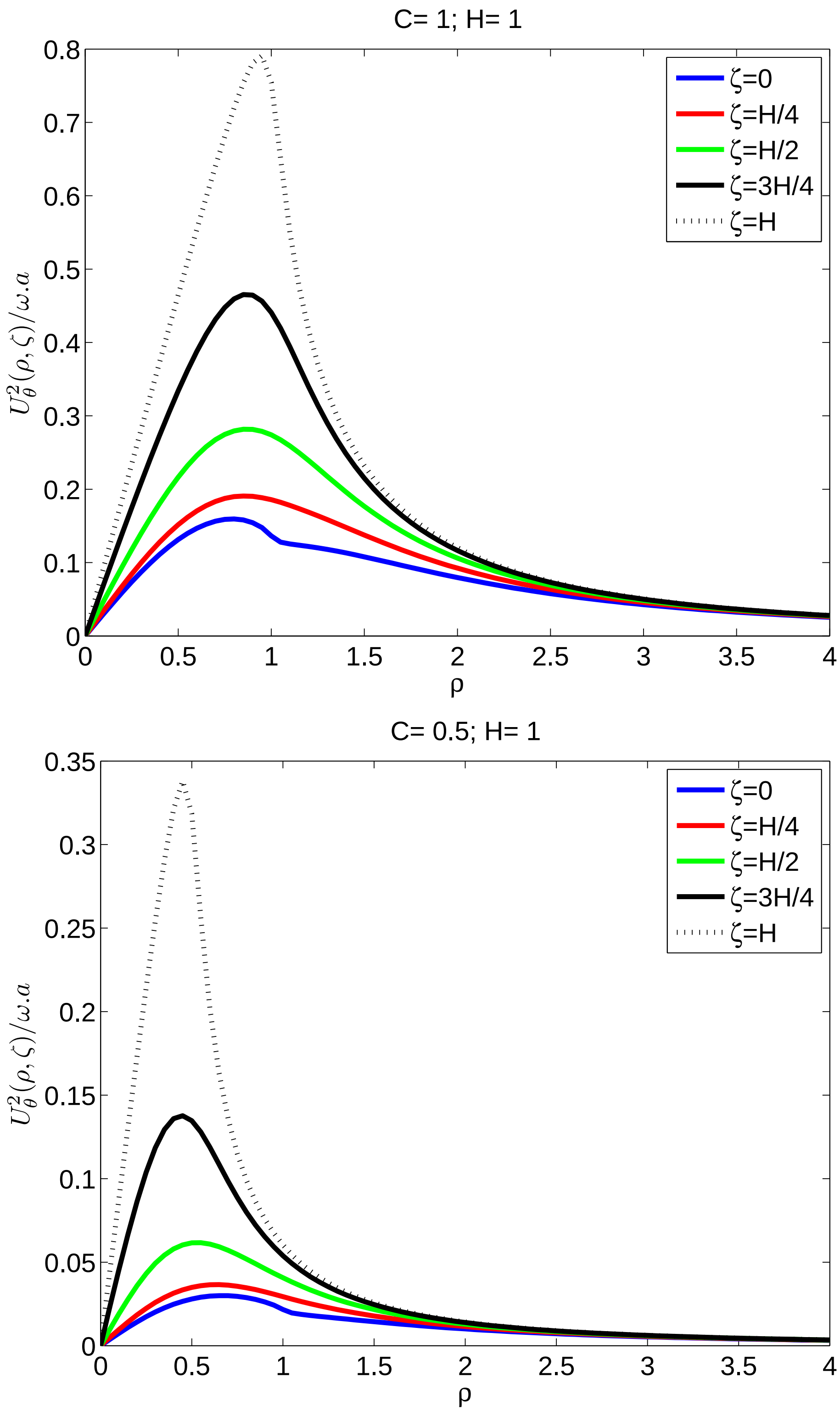


FIGURE 5 – Tangential displacement u_θ^2 versus ρ for various ζ, $0 \leq z \leq h$

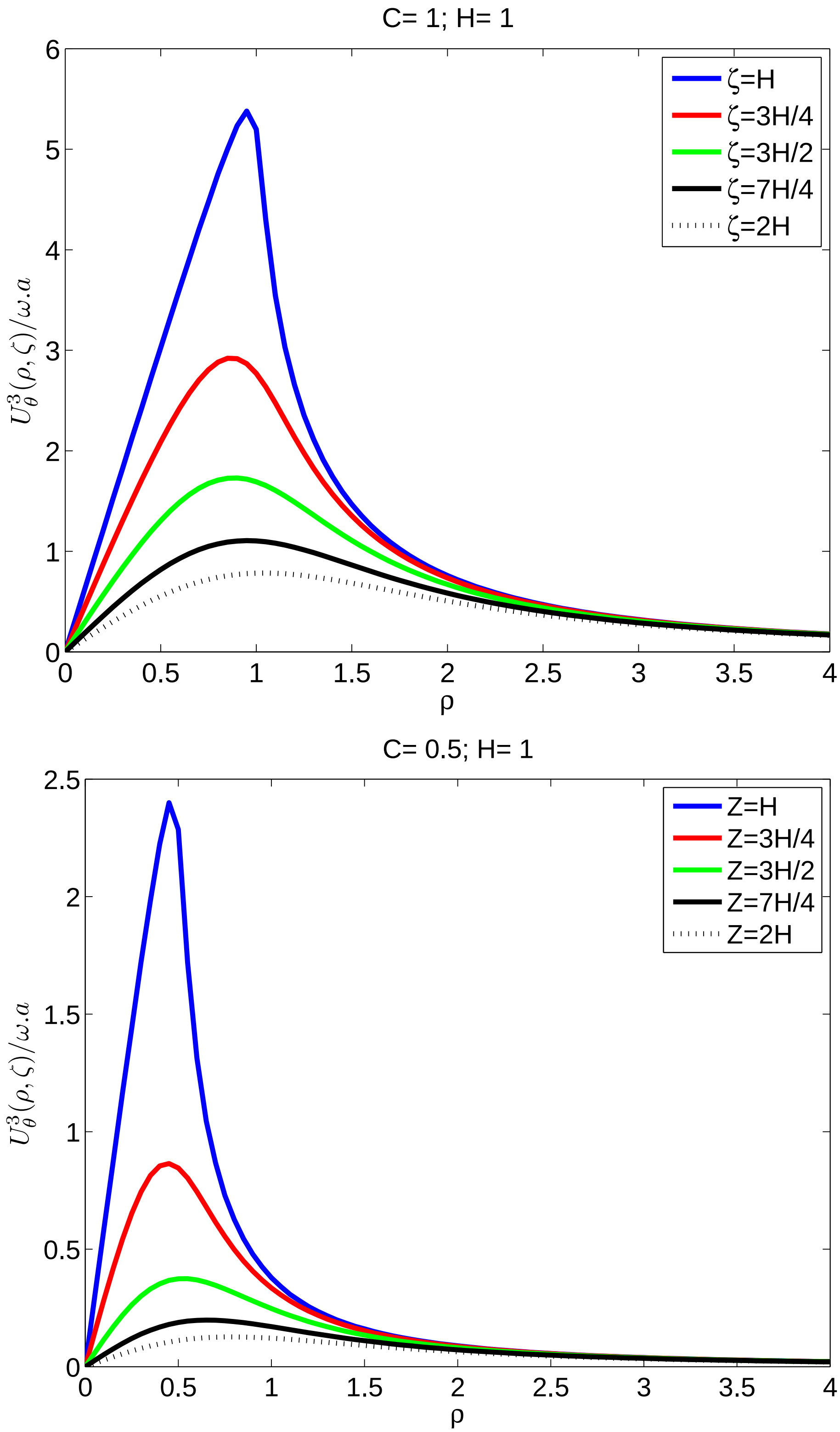


FIGURE 6 – Tangential displacement u_θ^3 versus ρ for various ζ, $z \geq h$

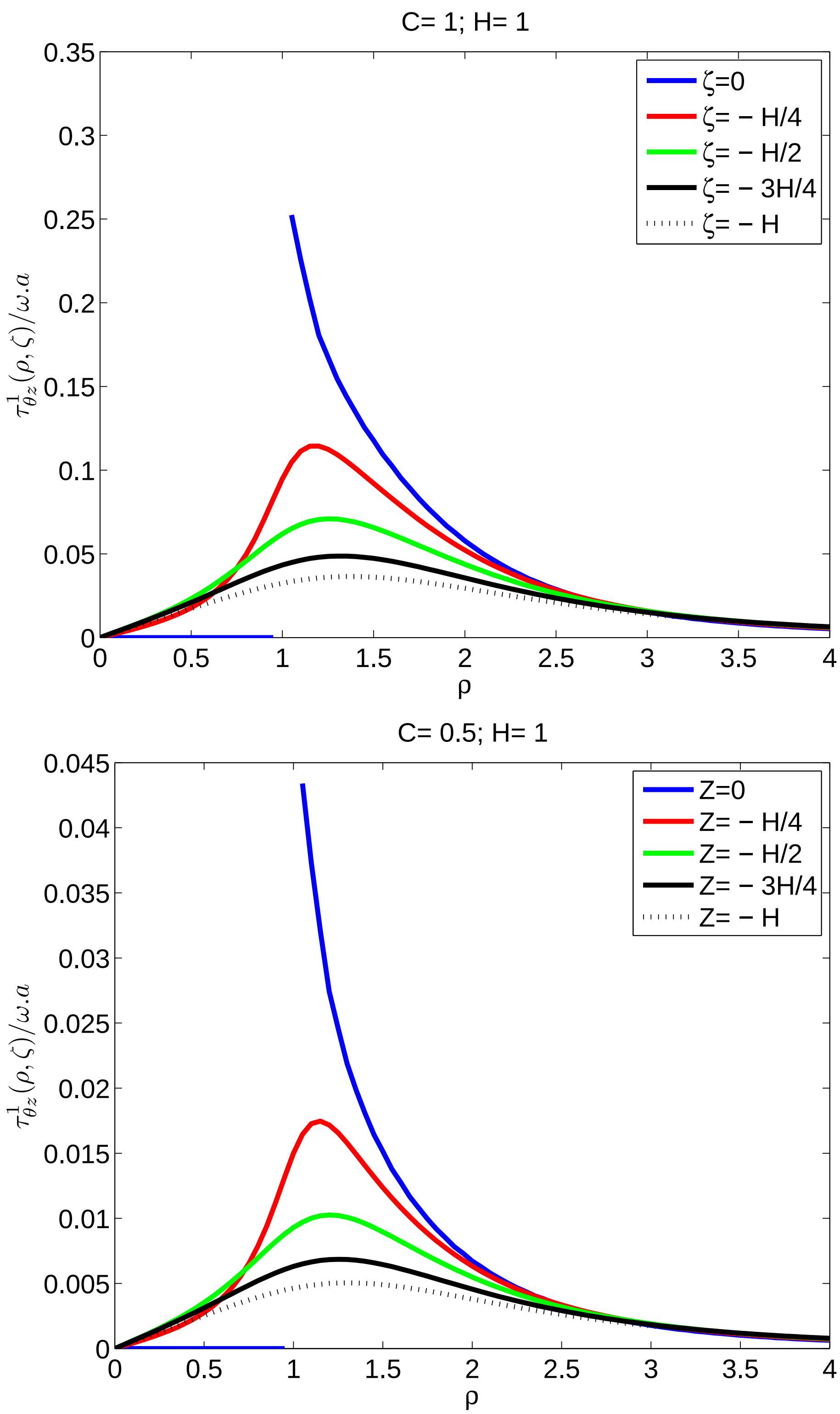


FIGURE 7 – Shear stress $\tau^1_{\theta z}$ versus ρ for various ζ, $z \leq 0$

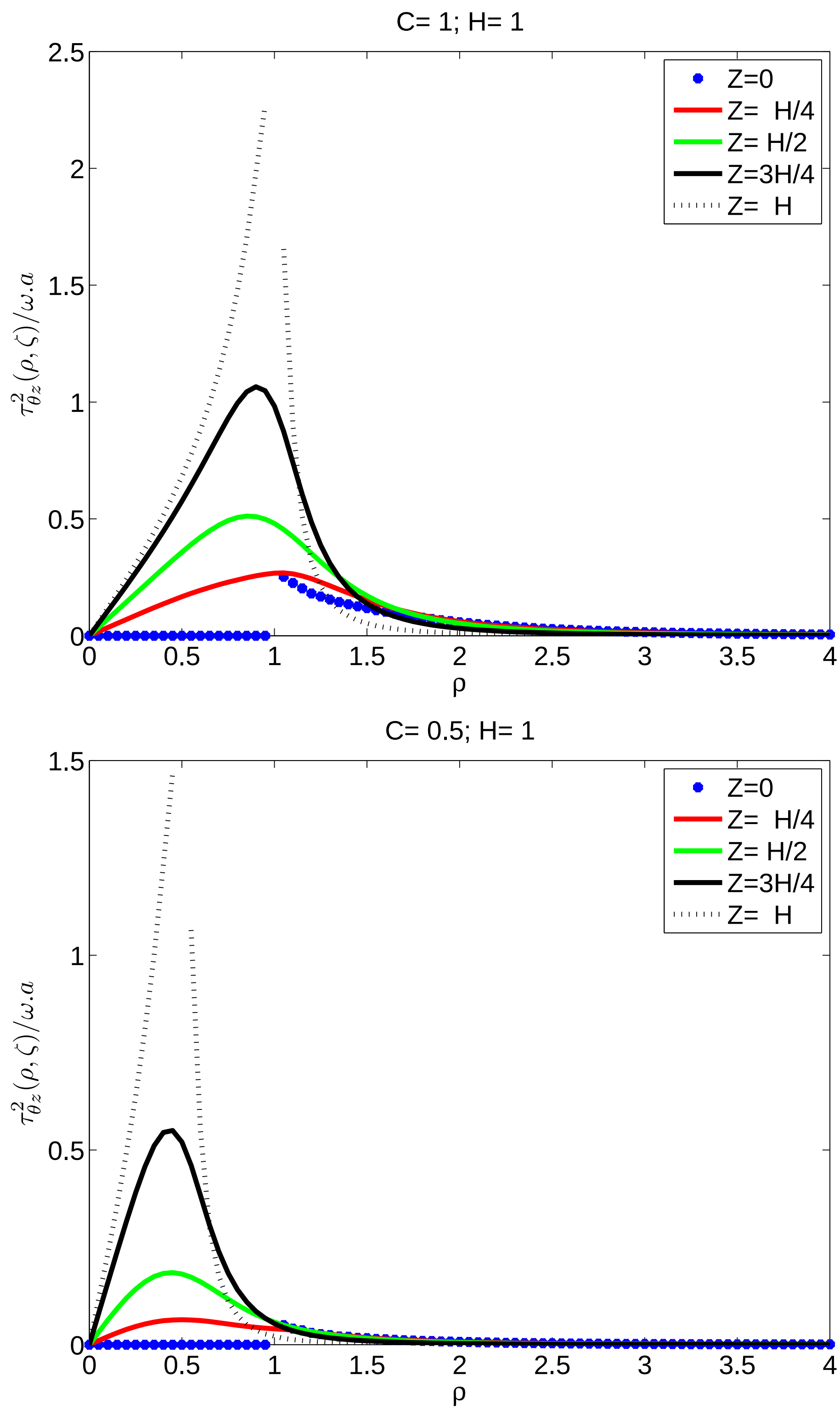
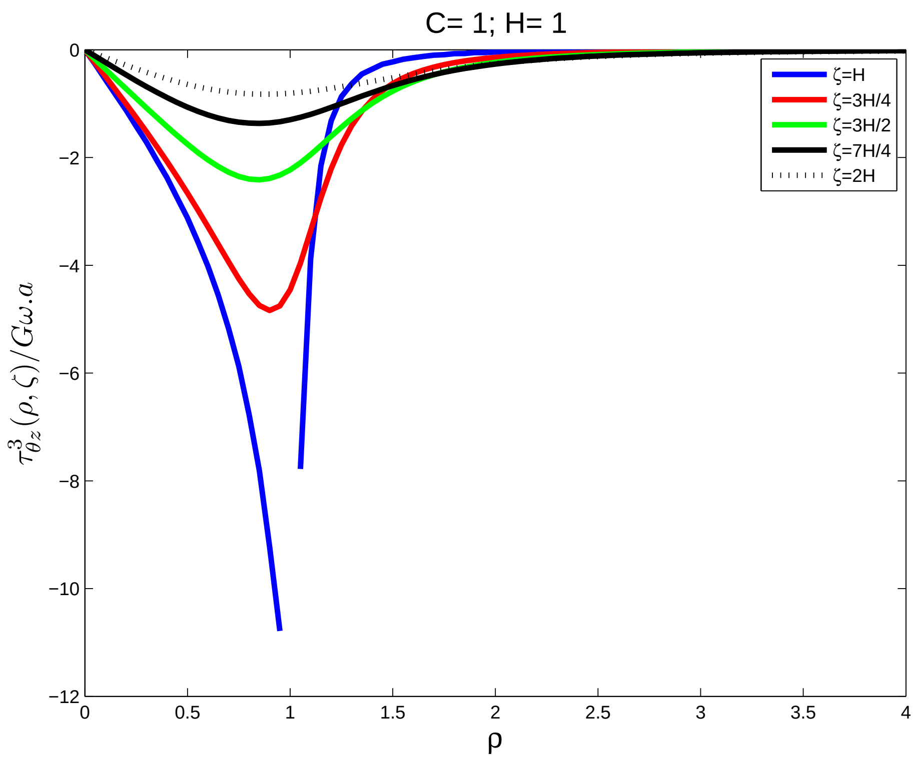


FIGURE 8 – Shear stress $\tau^2_{\theta z}$ versus ρ for various ζ, $0 \leq z \leq h$



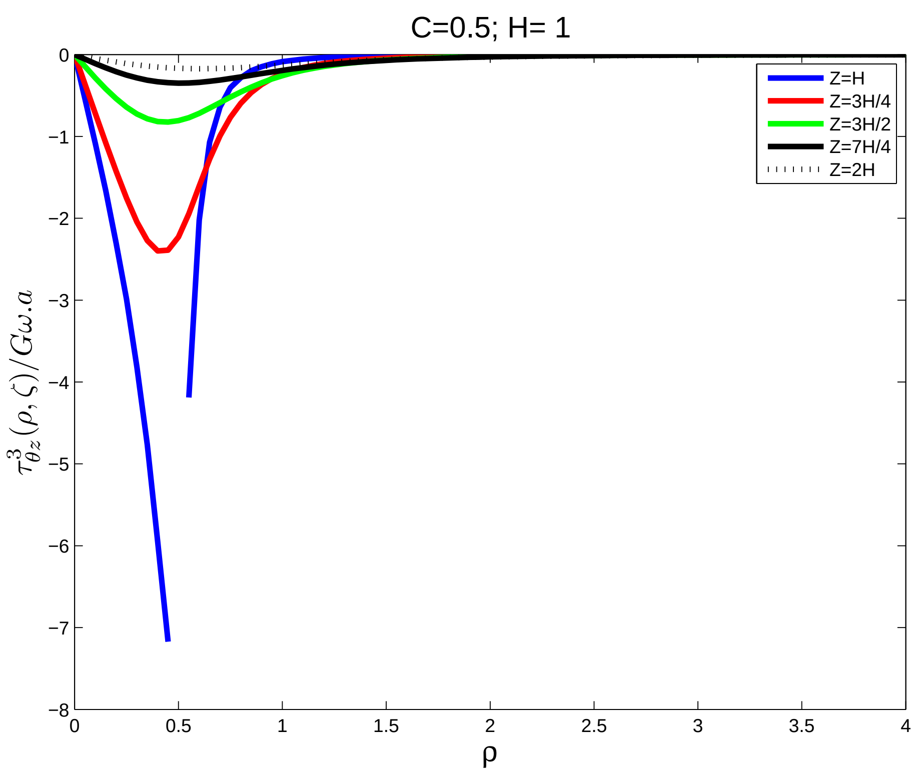


FIGURE 9 – Shear stress $\tau^3_{\theta z}$ versus ρ for various ζ, $z \geq h$

for some dimensionless parameters. The numerical results show that the discontinuities around the crack and the inclusion cause a large increase in the stresses which decay with distance from the disc-loaded. Furthermore, it can be seen the dependence of the stress intensity factor on the disc size and the distance between the crack and the rigid inclusion.

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