

STABILITY BOUNDS COMPARISON IN THE $(R, S, L_N Q)$ INVENTORY MODEL

Nedjma Aiane, Fazia Rahmoune and Djamil Aïssani

Research Unit LaMOS (Modeling and Optimization of Systems),
Faculty of Exact Sciences,
University of Bejaia, Bejaia, 06000, Algeria
E-mail: aianenedjma@hotmail.com
E-mail: rahmouneaoudia@gmail.com
E-mail: lamos_bejaia@hotmail.com

ABSTRACT

In this work, we study the sensitivity of a periodic review inventory system under the $(R, s, l_n Q)$ control policy and instantaneous replenishment of non-perishable items. We mainly choose the strong stability method and the absolute stability method to evaluate the approximation error of some characteristics due to the perturbation (demand rate, demand law). After having found the transition matrix of the Markov chain describing the studied model, we continue with a numerical study where we explore the effect of each perturbation of the $(R, s, l_n Q)$ model on its performance measures (average stock and total cost) and we end with a comparative study of the results established by the two methods.

1. INTRODUCTION

Stochastic inventory models where random influences can be taken into account remain among the most studied by operations research specialists. The control of the latter must be well enough managed to cope with all contingencies related to the randomness. Therefore, sensitivity analysis of stochastic inventory systems is imperative.

In this context, several research works have been carried out and different methods of analyzing these systems have been applied Aarya (2018), Molamohamadi et al (2020), Rabta and Aïssani (2005). As these models are very complicated, their resolution can only be approximated using different approximation methods based on perturbation theory. The analysis of a model via this theory aims to answer the following question : Can a real (complex) system be replaced by a more or less simple system whose characteristics are known, and under what conditions is this approximation possible? Aïssani and kartashov (1983), Rahmoune and Aïssani (2008), Rahmoune and Aïssani (2014).

From this point of view, several stability methods have been developed, including the strong stability method and the absolute stability method. Both methods are applied to study the sensitivity of models to perturbations in their parameters. In Rabta and Aïssani (2005), the authors prove, for the first time, the applicability of the strong stability method to the (R, s, S) type inventory model, where the perturbation only concerns the parameter describing the demand distribution. In Rabta (2017) and Rabta (2019), the author opted for the absolute stability method using the ergodicity coefficient and the generalized inverse of the transition matrix to study the sensitivity of the same inventory model. In this paper, after generalizing the control policy (R, s, Q)

by $(R, s, l_n Q)$, we study the sensitivity of the model to perturbation occurring either at the level of demand or at the level of its parameter, via the absolute stability method using the ergodicity coefficient principle. The results obtained by this last method were compared with those established via the strong stability method previously Aiane et al (2021). The comparison of the two approaches, with respect to the approximation error on different characteristics, namely the average stock as well as the total cost, allowed us to establish a statement on the characterization of the perturbation with respect to the set of variations of the considered parameters (controlled or uncontrolled parameters).

All concepts and notations not defined here can be found in (Rahmoune and Aïssani (2014), Kartashov (1996), Aissani and kartashov (1983), Ipsen and Meyer (1994), Rabta (2017), Seneta (1984), Seneta (1988), Seneta (1993)). The rest of the paper is organised as follows. In section 2, we present the description of the mathematical model where the corresponding Markov chain has been defined and the corresponding transition kernel. In Section 3, we briefly present the theoretical results of the strong v-stability method to the Markov chain describing the inventory model $(R, s, l_n Q)$ already established in Aiane et al (2018) and Aiane et al (2021). In Section 4, we apply the absolute stability method, using the ergodicity coefficient principle, to the Markov chain describing the inventory model $(R, s, l_n Q)$, after perturbing the parameter and the demand law, respectively. In section 5, we present the approximation bounds of some model performance measures. Section 6 is reserved for the numerical examples established, followed by a comparison of the results established by the two methods. We end with a conclusion and an appendix where the numerical results established via the strong stability method are presented.

2. THE MODEL DESCRIPTION

Consider the single-item, single-echelon periodic inventory model with $(R, s, l_n Q)$ policy of non-perishable items. According to this policy, the stock level is inspected every R units of time and an order is issued if the stock position is less than or equals to s . The order size is equal to the smallest multiple of Q such that the inventory level is raised above s , we assume that the lead time is zero, the excess demand is lost and the total demand is a discrete random variable ξ_n . We assume that the random variables $\xi_n, n \geq 1$, are independent and identically distributed of common probabilities given by $a_k = P(\xi_n = k)$.

The on-hand inventory level X_{n+1} at the end of period $n + 1$ is given by :

$$X_{n+1} = \begin{cases} (X_n - \xi_{n+1})^+ & \text{if } X_n > s; \\ (X_n + l_n Q - \xi_{n+1})^+ & \text{if } X_n \leq s. \end{cases}$$

where X_n is the on-hand inventory level at the end of period n , ξ_n is the total demand during the period n , $(A)^+ = \max(A, 0)$, l_n and $l_n \in N^*$ is the first integer verifying $X_n + l_n Q > s$ such as $l_n = \lceil \frac{s - X_n}{Q} \rceil + 1$. $\{X_n, n \geq 0\}$ is a homogeneous Markov chain with finite state space $E = \{0, 1, \dots, s, s + 1, \dots, s + Q\}$, the expression of the transition matrix is given by :

$$P_{ij} = \begin{cases} \sum_{k=i+l_n Q}^{\infty} a_k & \text{If } i \leq s \text{ and } j = 0; \\ a_{i-j+l_n Q} & \text{If } 0 < i \leq s \text{ and } 0 < j \leq s; \\ b_{ij} & \text{If } 0 < i \leq s \text{ and } j > s; \\ \sum_{k=i}^{\infty} a_k & \text{If } i > s \text{ and } j = 0; \\ a_{i-j} & \text{If } s + 1 \leq i \leq s + Q \text{ and } 0 < j \leq i; \\ 0 & \text{If } s + 1 \leq i \leq s + Q \text{ and } j > i; \end{cases}$$

where

$$b_{ij} = \begin{cases} a_{i-j+l_n Q} & \text{If } 0 < i \leq s, \quad j > s \text{ and } i - j + l_n Q \geq 0; \\ 0 & \text{If } 0 < i \leq s, \quad j > s \text{ and } i - j + l_n Q < 0; \end{cases}$$

3. STRONG ν -STABILITY OF THE $(R, S, L_N Q)$ INVENTORY MODEL

Denote by Γ the considered inventory model. Let $X = \{X_n, n \geq 0\}$ be the Markov chain representing the inventory level at date $t_n = nR$ in Γ . Let Γ' be another inventory model with the same structure as Γ , where we perturb the demand distribution of $\{\xi_n, n \geq 0\}$ expressed by : $a'_k = P(\xi'_n = k), k = 0, 1, \dots$
 Let $X' = \{X'_n, n \geq 0\}$ be the Markov chain representing the inventory level at date $t_n = nR$ in Γ' . Denote by P and Q the transition matrices of one-step of the Markov chains X and X' , respectively.

To prove the strong ν -stability of the Markov chain X_n we check the conditions of theorem done in Aissani and kartashov (1983), we choose a test function $\nu(k) = \beta^k$, where $\beta > 1$, the measurable function :

$$h_i = \begin{cases} 1 & \text{if } i=0; \\ 0 & \text{if } 0 < i \leq s+Q; \end{cases}$$

and the measure : $\alpha_j = P_{0j}, \forall j \in E$. Thus, we verify the strong ν -stability conditions, namely : $\pi h > 0, \alpha \mathbb{1} = 1, \alpha h > 0$ and that $T = P - h\alpha$ is a non negative matrix for witch we must find some constant $\rho < 1$ such as $T\nu(k) \leq \rho\nu(k)$ for all $k \in E$ and verify that $\|P\|_\nu < \infty$.

3.1. Quantitative estimate

In order to estimate the difference between the stationary distributions of Markov chains X_n and X'_n , we have first to estimate the difference between the transition kernel P and Q , then the deviation $\|P - Q\|_\nu$ can be calculated by :

$$\|P - Q\|_\nu = \sup_{k \in \{0, \dots, s+Q\}} \frac{1}{\beta^k} \sum_{j=0}^{s+Q} |P_{kj} - Q_{kj}| \beta^j \quad (1)$$

it follows that

$$\|P - Q\|_\nu = \sum_{i=s+Q}^{\infty} |a_i - a'_i| + \sum_{i=0}^{s+Q-1} |a_i - a'_i| \beta^{s+Q-i} \quad (2)$$

Now, we can estimate the gap between the stationary distributions π and π' of the Markov chains X_n and X'_n respectively. for this we use the following theorem done in Kartashov (1996) :

Theorem 1. For $\Delta = Q - P$ verifying the condition

$$\|\Delta\|_\nu < C^{-1}(1 - \rho) \quad (3)$$

we have :

$$\|\pi' - \pi\|_\nu \leq \|\Delta\|_\nu \|\pi\|_\nu C(1 - \rho - C\|\Delta\|_\nu)^{-1}; \quad (4)$$

where

$$C = 1 + \|\mathbb{1}\|_\nu \|\pi\|_\nu \quad (5)$$

and

$$\|\pi\|_\nu \leq (\alpha\nu)(1 - \rho)^{-1}(\pi h). \quad (6)$$

In order to obtain the estimate of $\|\pi' - \pi\|_\nu$, let us first estimate $\|\pi\|_\nu$ and $\|\mathbb{1}\|_\nu$, where $\mathbb{1}$ is a function identically equal to unity, secondly calculate the value of the constant C given in the theorem and finally replace the all in (4).

where

$$\alpha v = \sum_{i=s+Q}^{\infty} a_i + \sum_{i=0}^{s-1+Q} a_i \beta^{s+Q-i}$$

and

$$\pi h = \sum_{i=0}^{s+Q} \pi_i h(i) = \sum_{i=0}^s \pi_i$$

Finally, we obtain

$$\|\pi\|_v \leq \left(\sum_{i=s+Q}^{\infty} a_i + \sum_{i=0}^{s-1+Q} a_i \beta^{s+Q-i} \right) (1-\rho)^{-1} \left(\sum_{i=0}^s \pi_i \right)$$

The estimation of $\|\mathbb{1}\|$ is

$$\|\mathbb{1}\|_v = \sup_{j \in \{0, \dots, s+Q\}} \frac{1}{v(j)} = \sup_{j \in \{0, \dots, s+Q\}} \frac{1}{\beta^j} = 1. \quad (7)$$

where

$$C = 1 + \|\pi\|_v \quad (8)$$

And $\|\Delta\|_v$ is given by (2) and ρ is

$$\rho = \frac{\sum_{i=s+1}^{\infty} a_i}{\beta^{s+1}} + \sum_{i=0}^s a_i \beta^{-i} \quad (9)$$

which is smaller than 1 for all $\beta > 1$.

4. ABSOLUTE STABILITY OF THE $(R, S, L_N Q)$ INVENTORY MODEL

The absolute stability method has been used by several authors to obtain bounds on the $\|\cdot\|_1$ norm of the stationary vector of an irreducible discrete Markov chain, using different quantities, as an example we cite the use of the fundamental matrix by Schweitzer (1968), the g-inverses by Hunter (1982), the ergodicity coefficient by Seneta (1984) and the inverse group by Ipsen and Meyer (1994). All these methods consist in using techniques of linear algebra and matrix computation. In the following, we apply the absolute stability method using the ergodicity coefficient to study the sensitivity of the $(R, s, l_n Q)$ inventory model to perturbations in the demand process. The latter has already been applied to the (R, s, S) model by Rabta (2017).

For the $(R, s, l_n Q)$ model described above, in order to estimate the difference between the stationary distributions of the Markov chains X_n and X'_n , we use the formula given by Eq.(10)

$$\|\pi' - \pi\| \leq K \|\Delta\|_1 \quad (10)$$

Where : $\|\Delta\|_1$ can be found by replacing β by 1 in the formula (1).

$$K = \frac{1}{1 - \tau(P)} \quad (11)$$

Such that $\tau(P)$ is the ergodicity coefficient of the perturbation matrix P given by (12) :

$$\tau(P) = \max_{u^T e=0, \|u^T\| < 1} \|u^T P\|. \quad (12)$$

Where : $e^T = (1, 1, \dots, 1)$ is the vector of all ones.

Seneta (1984) gives an explicit expression of this quantity :

$$\tau(P) = \frac{1}{2} \max_{i,j} \sum_{k=0}^{s+Q} |p_{ik} - p_{jk}| \quad (13)$$

This expression will be very useful in section 5, to estimate the error on the performance measures of the model $(R, s, l_n Q)$ via the absolute stability method which is based on the principle of the ergodicity coefficient.

5. THE APPROXIMATION BOUNDS OF SOME MODEL PERFORMANCE MEASURES

The perturbation bounds found previously via the two methods (strong stability and absolute stability using the ergodicity coefficient) will be used to estimate the error on the following performance measures : the average stock and the total cost. The estimation of the error on the average stock is given by the formula (14)

$$|\bar{X} - \bar{X}'| \leq \|\pi' - \pi\|_v \frac{s+Q}{2}. \quad (14)$$

where : \bar{X} and \bar{X}' represent the average stocks in the two systems respectively (before and after perturbation), given by :

$$\bar{X} = \sum_{i=0}^{s+Q} i\pi_i; \quad (15)$$

and

$$\bar{X}' = \sum_{i=0}^{s+Q} i\pi'_i. \quad (16)$$

The expression of the total cost C_T of the considered model is given by Eq. (17)

$$\begin{aligned} C_T = & \sum_{i=0}^s \pi_i (C_1(i + l_n Q) + C_2 \sum_{k=i+l_n Q+1}^{\infty} (k - (i + l_n Q)) a_k C_3 + C_4 \times l_n Q) \\ & + \sum_{i=s+1}^{s+Q} \pi_i (C_1 \times i + C_2 \sum_{k=i+1}^{\infty} (k - i) a_k). \end{aligned} \quad (17)$$

Remark : Details of the computation of the formulae given by Eq.(14) and Eq.(17) respectively are given in Aiane et al (2021).

6. NUMERICAL ILLUSTRATIONS

In this section, we present some numerical examples to illustrate the performance of the approximation methods used. This numerical study has been carried out in two main parts, depending on the variation of the controlled parameters (s and Q) of the model studied. Each part is in turn carried out in two other parts according to the perturbed parameter (perturbation of the demand law parameter, perturbation of the demand law).

In each part, we consider the $(R, s, l_n Q)$ inventory model with a Poisson distribution of demand with parameter λ when the perturbation concerns the parameter of the demand law, and when the perturbation concerns the the demand law, we choose the binomial distribution with

parameters n and p . After that, we study the effect of these perturbation on the following performance measures : the stationary distributions, the average stock as well as the total cost of the considered model.

The inventory costs considered in each part are the following :

- holding cost $C_1 = 5$ (per unit in stock and per period) ;
- shortage cost $C_2 = 10$ (per unit of sales lost)
- fixed order cost $C_3 = 10$ (per order) ;
- unit (variable) order cost $C_4 = 5$ (per unit ordered).

Part 1 : In this part, the model input parameters (R, s, l_n, Q) are set to : $R = 1, s = 5$ and $Q = 2$.

6.1. Perturbation of the demand rate

The perturbation considered concerns the arrival rate, which may be caused, in reality, by the error due to the estimation of the parameter in question. For different values of λ ($\lambda = 5, \lambda = 10, \lambda = 15, \lambda = 20$), we posit $\lambda = \lambda + \varepsilon$, where ε characterizes the perturbation.

Table 1 summarizes the different values of deviation of the transition kernels $\|P - Q\|_1$, deviation of the stationary distributions $\|\pi' - \pi\|_1$, the estimation of the error on the average stock $|\bar{X} - \bar{X}'|$ as well as the total inventory cost before and after perturbation for each value of considered λ and ε .

We denote by C_T the total cost before perturbation and by \bar{C}_T the total cost after perturbation.

λ	λ'	$\ P - Q\ _1$	$\tau(P)$	$\ \pi' - \pi\ _1$	$ \bar{X} - \bar{X}' $	C_T	\bar{C}_T
5	5.01	0.0035	0.1755	0.0043	0.0064	68.6952	78.7606
10	10.01	0.0013	0.0631	0.0013	0.0020	110.5803	110.6766
15	15.01	9.65×10^{-5}	0.0048	9.6969×10^{-5}	1.4545×10^{-4}	160.0196	160.1195
15	15.1	9.3939×10^{-4}	0.0048	9.4396×10^{-4}	0.0014	160.0196	161.0182
15	15.3	0.0027	0.0048	0.0027	0.0040	160.0196	163.0158
15	15.5	0.0042	0.0048	0.0042	0.0063	160.0196	165.0137
20	20.01	3.6515×10^{-6}	1.8321×10^{-4}	3.6515×10^{-6}	5.4782×10^{-6}	210.0005	210.1005
20	20.1	3.5389×10^{-5}	1.8321×10^{-4}	3.5395×10^{-5}	5.3093×10^{-5}	210.0005	211.004
20	20.3	9.9132×10^{-5}	1.8321×10^{-4}	9.9150×10^{-5}	1.4873×10^{-4}	210.0005	213.004
20	20.5	1.5450×10^{-4}	1.8321×10^{-4}	1.5453×10^{-4}	2.3179×10^{-4}	210.0005	215.0003

TABLE 1 – The different values of the deviation of the transition matrix $\|P - Q\|_1$, the approximation error $\|\pi' - \pi\|_1$, the estimation of the error on the average stock $|\bar{X} - \bar{X}'|$ and the total inventory cost before and after perturbation for each value of considered λ and ε .

6.2. Perturbation of the demand distribution

The perturbation considered concerns the demand law, which can be caused, in reality, by an error during the test of the laws adjustment. For the perturbation of the demand law, we choose the binomial distribution with parameters n and p . We perturb the parameter n by taking $n = \frac{\lambda + \varepsilon}{p}$. For example, in the table 2, for the following parameters : $\lambda = 18, \varepsilon = -2$ and $p = 0.1$ we will thus have $n = 160$ and $n \times p = 16$, which is close to $\lambda = 18$. Consequently the Binomial distribution with parameters n and p can be approximated by the Poisson distribution with the

parameter λ .

Table 2 summarizes the different values of deviation of the transition kernels $\|P - Q\|_1$, deviation of the stationary distributions $\|\pi' - \pi\|_1$, the estimation of the error on the average stock $|\bar{X} - \bar{X}'|$ as well as the total inventory cost before and after perturbation for each value of considered n and ε .

ε	n	$n \times p$	$\tau(P)$	$\ P - Q\ _1$	$\ \pi' - \pi\ _1$	$ \bar{X} - \bar{X}' $	C_T	\bar{C}_T
-2	160	16	7.1945×10^{-4}	0.0035	0.0035	0.0053	190.0021	170.0061
-1	170	17	7.1945×10^{-4}	6.7587×10^{-4}	6.7635×10^{-4}	0.0010	190.0021	180.0028
0.5	185	18.5	7.1945×10^{-4}	0.0012	0.0012	0.0018	190.0021	195.0008
1	190	19	7.1945×10^{-4}	0.0015	0.0015	0.0022	190.0021	200.0006
2	200	20	7.1945×10^{-4}	0.0018	0.0018	0.0027	190.0021	210.0002
3	210	21	7.1945×10^{-4}	0.0020	0.0020	0.0029	190.0021	220.0001

TABLE 2 – The different values of the deviation of the transition matrix $\|P - Q\|_1$, the approximation error $\|\pi' - \pi\|_1$, the estimation of the error on the average stock $|\bar{X} - \bar{X}'|$ and the total inventory cost before and after perturbation for each value of considered n and ε .

Part 2 : In this second part, the model input parameters (R, s, l_n, Q) are set to : $R = 1, s = 100$ and $Q = 95$.

In what follows, we carry out exactly the same procedure established in Part 1.

6.3. Perturbation of the demand rate

Table 3 summarizes the different values of deviation of the transition kernels $\|P - Q\|_1$, deviation of the stationary distributions $\|\pi' - \pi\|_1$, the estimation of the error on the average stock $|\bar{X} - \bar{X}'|$ as well as the total inventory cost before and after perturbation for each value of considered λ and ε .

Remember that C_T the total cost before perturbation and by \bar{C}_T the total cost after perturbation.

λ	λ'	$\ P - Q\ _1$	$\ \pi' - \pi\ _1$	$ \bar{X} - \bar{X}' $	C_T	\bar{C}_T
65	65.01	9.8839×10^{-4}	2.4468×10^5	1.1745×10^7	1.0717×10^3	1.0716×10^3
65	64.99	9.8846×10^{-4}	2.4470×10^5	1.1746×10^7	1.0717×10^3	1.0717×10^3
64	64.01	9.9606×10^{-4}	3.3123×10^5	1.5899×10^7	1.0726×10^3	1.0727×10^3
64	63.99	9.9614×10^{-4}	3.3126×10^5	1.59×10^7	1.0726×10^3	1.0725×10^3
63	63.01	0.0010	4.5007×10^5	2.1603×10^7	1.0570×10^3	1.0571×10^3
63	62.99	0.0010	4.5011×10^5	2.1605×10^7	1.0570×10^3	1.0568×10^3
62	62.01	0.0010	6.1344×10^5	2.9445×10^7	1.0534×10^3	1.0534×10^3
62	61.99	0.0010	6.1349×10^5	2.9447×10^7	1.0534×10^3	1.0535×10^3

TABLE 3 – The different values of the deviation of the transition matrix $\|P - Q\|_1$, the approximation error $\|\pi' - \pi\|_1$, the estimation of the error on the average stock $|\bar{X} - \bar{X}'|$ and the total inventory cost before and after perturbation for each value of considered λ and ε .

Remark : Regarding the perturbation of the law, while considering the same input parameters of the model ($R = 1, s = 100, Q = 95$), the results obtained via both methods (strong stability and absolute stability using the ergodicity coefficient) are not satisfactory, however, The strong stability method is still applicable but for the absolute stability method using the ergodicity coefficient, we have $\tau(P)$ approaches more then to the value 1, which leads to bad results and once tau $\tau(P)$ is higher than 1 the absolute stability method using the ergodicity coefficient is not applicable anymore.

6.4. Comparison and interpretation of results

The numerical results obtained via the two approximation methods show : that when the controllable parameters of the inventory system $(R, s, I_n Q)$ namely s and Q are small. The results obtained via the absolute stability method in terms of the ergodicity coefficient are better than those obtained via the strong stability method (these numerical results are shown in the Appendix), as the error estimated by the absolute stability method based on the ergodicity coefficient is small compared to that estimated by the strong stability method. This leads to a small error on the performance of the system (the average stock, the total cost). This result remains valid for the two perturbations considered (Perturbation of the demand rate, perturbation of the demand law).

When the input parameters of the inventory system $(R, s, I_n Q)$ are large enough and the perturbation concerns the parameter describing the demand law (demand rate), we notice that the results obtained via the strong stability method are better than those obtained via the absolute stability method using the ergodicity coefficient.

When the perturbation concerns the demand law with input parameters of the $(R, s, I_n Q)$ inventory system are large enough, the numerical results obtained via the strong stability method are not satisfactory, but the latter is still applicable.

7. CONCLUSION

In this work, we have applied the absolute stability method using the ergodicity coefficient to study the sensitivity of the single-item, single-echelon periodic inventory model with $(R, s, I_n Q)$

policy of non-perishable items, and we have recalled some results obtained when applying the strong stability method to the same inventory model.

The numerical results obtained via the two approximation methods illustrate the effect of the two perturbations considered (perturbation of the arrival rate, perturbation of the demand law) on the stationary distributions and the following characteristics : the average stock and the total cost of the $(R, s, l_n Q)$ inventory model.

From a comparative study of the results obtained via the two methods (absolute stability in terms of the ergodicity coefficient and strong stability), we note that, the method of strong stability gives better results when the input parameters of the system are increasingly large and as in reality, the stocks constituted by the companies are rather important in order to decrease the risk of stock shortage. This encourages us to adapt the strong stability method to estimate the errors due to the approximation of a real system by another ideal system which is close to it in a certain sense.

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Appendix

• **The numerical results obtained using the strong stability method to study the sensitivity of the single-item, single-echelon periodic inventory model with $(R, s, l_n Q)$ policy of non-perishable items :**

Part 1 : In this part, the model input parameters $(R, s, l_n Q)$ are set to : $R = 1, s = 5$ and $Q = 2$.

Table 4 summarizes the different values of deviation of the stationary distributions $\|\pi' - \pi\|_v$, the estimation of the error on the average stock $|\bar{X} - \bar{X}'|$ as well as the total inventory cost before and after perturbation for each value of considered λ and ε , when the parameter of the demand law is perturbed.

λ	λ'	$\ \pi' - \pi\ _v$	$ \bar{X} - \bar{X}' $	C_T	\bar{C}_T
5	5.01	0.0295	0.1033	68.6952	68.760
10	10.01	0.0436	0.1527	110.5803	110.6766
15	15.01	0.0026	0.0092	160.0196	160.1195
15	15.1	0.0258	0.0906	160.0196	161.0182
15	15.3	0.0074	0.2617	160.0196	163.0158
15	15.5	0.1192	0.4191	160.0196	165.0137
20	20.01	0.0791×10^{-3}	2.7674×10^{-4}	210.0005	210.1005
20	20.1	0.7654×10^{-3}	0.0027	210.0005	211.004
20	20.3	0.0021	0.0075	210.0005	213.004
20	20.5	0.0033	0.0116	210.0005	215.0003

TABLE 4 – The different values of the deviation of the approximation error $\|\pi' - \pi\|_v$, the estimation of the error on the average stock $|\bar{X} - \bar{X}'|$ and the total inventory cost before and after perturbation for each value of considered λ and ε .

Table 5 summarizes the different values of deviation of the stationary distributions $\|\pi' - \pi\|$, the estimation of the error on the average stock $|\bar{X} - \bar{X}'|$ as well as the total inventory cost before and after perturbation for each value of considered λ and ε , when the law of the demand is perturbed.

Part 2 : In this part, the model input parameters $(R, s, l_n Q)$ are set to : $R = 1, s = 100$ and $Q = 95$.

Table 6 summarizes the different values of deviation of the stationary distributions $\|\pi' - \pi\|_v$, the estimation of the error on the average stock $|\bar{X} - \bar{X}'|$ as well as the total inventory cost before and after perturbation for each value of considered λ and ε , when the parameter of the demand law is perturbed.

ε	n	$n \times p$	$\ \pi' - \pi\ $	erstock	C_T	$\overline{C_T}$
-2	160	16	0.0861	0.3012	190.0021	170.0061
-1	170	17	0.0144	0.0504	190.0021	180.0028
0.5	185	18.5	0.0277	0.0971	190.0021	195.0008
1	190	19	0.0342	0.1197	190.0021	200.0006
2	200	20	0.0417	0.1458	190.0021	210.0002
3	210	21	0.0451	0.1580	190.0021	220.0001

TABLE 5 – The different values of the deviation of the approximation error $\|\pi' - \pi\|_v$, the estimation of the error on the average stock $|\overline{X} - \overline{X}'|$ and the total inventory cost before and after perturbation for each value of considered n and ε .

λ	λ'	$\ P - Q\ _v$	$\ \pi' - \pi\ _v$	erstock	C_T	$\overline{C_T}$
65	65.01	0.6412	6.2388×10^{-4}	0.0608	1.0717×10^3	1.0716×10^3
65	64.99	0.6416	6.2492×10^{-4}	0.0609	1.0717×10^3	1.0717×10^3
64	64.01	0.6770	4.0377×10^{-4}	1.5899×10^7	1.0726×10^3	1.0727×10^3
64	63.99	0.6774	4.0453×10^{-4}	0.0394	1.0726×10^3	1.0725×10^3
63	63.01	0.7148	2.8224×10^{-4}	0.0275	1.0570×10^3	1.0571×10^3
63	62.99	0.7152	2.8286×10^{-4}	0.0276	1.0570×10^3	1.0568×10^3
62	62.01	0.7549	2.0710×10^{-4}	0.0202	1.0534×10^3	1.0534×10^3
62	61.99	0.7553	2.0765×10^{-4}	0.0202	1.0534×10^3	1.0535×10^3

TABLE 6 – The different values of the deviation of the transition matrix $\|P - Q\|_v$ and the approximation error $\|\pi' - \pi\|_v$, the estimation of the error on the average stock $|\overline{X} - \overline{X}'|$ and the total inventory cost before and after perturbation for each value of considered λ and ε .