STOCHASTIC DIFFERENTIAL EQUATION IN IMAGE RESTORATION

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ABSTRACT

In this work, we propose to solve the problem of restoring images using stochastic differential equations (SDEs). In a recent work [5], we combined two approaches due to Barbu [1, 2] and Borkowski [3] and deducted that the SDEs are very promising in image restoration.The main idea here is to try to find a good choice for the drift and diffusion terms. In 1990 Perona-Malik (PM) [6] explored two functions to better handle contours of objects in images. In this proposed model, we exploit these functions in the SDE ; more precisely we try to fix the drift and diffusion terms according to PM-functions to improve our previous SDEs results. The obtained experimental results are very encouraging and competetive compare to other SDEs models.

1. INTRODUCTION

Recently, the use of stochastic differential equations in image restoration has significantly grown and evolved. Various models have been proposed to solve this problem, including the works by Barbu et al [1] and Borkowski et al [3, 4], unfortunately the first one identified the diffusion term to 1 and the second neglected the drift term.

In our previous work, we incorporated both terms (the drift and the diffusion) in the SDE and obtained encouraging results compare to models based on the Partial Differential Equations (PDEs). Here, we propose a new SDE model with different drift and diffusion terms, exploring PM-functions, in order to improve denoising with contour preservation.

2. PROPOSED MODEL

We consider the following model

$$\begin{cases} dX(t) = \underbrace{\mu(X(t))}_{drift}dt + \underbrace{\sigma(X(t))}_{diffusion}dW_t,\ X(0,x,y) = X_0(x,y) \in \mathbb{R}^2. \end{cases} \tag{1}$$

We suggest the drift and the diffusion terms to be the PM-funcions and Borkowski's diffusion and write

$$\mu = e^{-(\frac{\|\nabla X\|^2}{k^2})} \text{ or } \frac{1}{1+\frac{\|\nabla X\|^2}{k^2}},\ \text{and } \sigma(X(s)) = \begin{pmatrix} -\frac{(G_\gamma * I)_y(X_t)}{|\nabla(G_\gamma * I)(X_t)|} & 0 \\ \frac{(G_\gamma * I)_x(X_t)}{|\nabla(G_\gamma * I)(X_t)|} & 0 \end{pmatrix} \tag{2}$$

where $X(t) = (X_1(t), X_2(t))$ is the stochastic process and $W(t)$ represents a 2D-Brownian motion in a probability space$\{\Omega, , P\}$. The solution will be the stochastic process $X = (X_1(t,X_0), X_2(t,X_0))$ adapted to the natural filtration $(F_t)_{t\geq 0}$ on the probability space, which satisfies the following equation

$$X(t) = X_0 + \int_0^t \mu(X(s))ds + \int_0^t \sigma(X(s))dW_s. \tag{3}$$

The restored image is defined by

$$u(x,y) = E[u_0(X_T)] = \frac{1}{M}\sum_{i=1}^{M} u_0(X_T^m(\beta_i)) \tag{4}$$

with $u_0 : \mathbb{R}^2 \to \mathbb{R}$, the noisy image, $X^m(\beta_i)$ the approximation of trajectory for the stochastic process X and M is the number of Monte Carlo's iterations.

3. IMPLEMENTATION AND NUMERICAL RESULTS

3.1. Numerical scheme

By using the modified Euler's numerical scheme, we obtain

$$X_{t_{n+1}}^m(i,j) = X_{t_n}^m(i,j) + \mu(X_{t_n}^m(i,j))\Delta t + \sigma(X_{t_n}^m(i,j))dW_t, \tag{5}$$

where $t_n = nh$, $h = \frac{T}{m}$, $n = 0,1,..,m$ and we study two cases :

1. PM-functions as drift term and Borkowski's diffusion term, i.e.

$$\mu(X_{t_n}^m(i,j)) = \begin{pmatrix} e^{-\left(0.5\left(\frac{(X_1^m(i+1,j)-X_2^m(i-1,j))^2+(X_1^m(i,j+1)-X_2^m(i,j-1))^2}{k^2}\right)\right)} \\ \text{or} \\ \frac{1}{1+\left(0.5\left(\frac{(X_1^m(i+1,j)-X_2^m(i-1,j))^2+(X_1^m(i,j+1)-X_2^m(i,j-1))^2}{k^2}\right)\right)} \end{pmatrix} \tag{6}$$

and

$$\sigma(X_{t_n}^m(i,j)) = \begin{pmatrix} -\frac{((u(i,j+1)+u(i,j-1))*0.5)}{\sqrt{(u(i,j+1)+u(i,j-1))*0.5)^2+(u(i+1,j)+u(i-1,j))*0.5)^2}} & 0 \\ \frac{((u(i+1,j)+u(i-1,j))*0.5)}{\sqrt{(u(i,j+1)+u(i,j-1))*0.5)^2+(u(i+1,j)+u(i-1,j))*0.5)^2}} & 0 \end{pmatrix}. \tag{7}$$

with $(G_\gamma * I)_y = u_y$=$((u(i,j+1)+u(i,j-1))*0.5)$ and $(G_\gamma * I)_x = u_x = (u(i+1,j)+u(i-1,j))*0.5)$. Note that the space steps $dx = dy = 1$.

2. Barbu's drift term [1] and PM-functions (6) as diffusion term.

3.2. Results and comments

From the obtained results in Table 1. and Figure 1., we note that the proposed model provides good results in noise removal and important image features perserving (such as texture, curvature, ect..), compare to the combined Barbu-Borkowski SDE (Barbu's drift and Borkowski's diffusion terms) [5]. Both cases are more or less similar ; but results related to the second case, where the drift and the diffusion terms are chosen to be the Barbu's drift and PM-functions, respectively, give a slight improvement to the first case.

The numerical results shown in Table 1. and Figure 1, are presented on a grey-scale image altered by Gaussian white noise of zero mean and variance $V = 0.01$ and 0.1, in terms of the PSNR (Peak Signal to Noise Ratio) and SSIM (Structure Similarity Index Measure)

4. CONCLUSION

In this work, we proposed a stochastic based model for image restoration. The model uses both drift and diffusion terms in the SDE for noise removal and without losing significant features or creating false ones. The drift and the diffusion terms have been reformulated in order to adapt the stochastic process to the complexity of the geometric structures. From the obtained numerical results, we deduct that our proposed model tends to be qualitatively more efficient compare to the combined Barbu-Borkowski SDE one [5], this is not only visually as shown in Figure 1, but also by the means of the PSNR and SSIM as reported in Table 1. It has to be pointed out that the proposed model case 2 is intuitively more realistic.
Other numerical experimentations are under consideration.

Table 1. A Comparison of results for image restoration

	Proposed Model (case 1)		Combined Model		Proposed Model (case 2)	
Variance	PSNR	SSIM	PSNR	SSIM	PSNR	SSIM
0.01	**35.7052**	**0.9781**	30.9638	0.8645	**38.6025**	**0.9865**
0.1	**23.8868**	**0.7758**	23.0485	0.5656	**24.5423**	**0.7926**

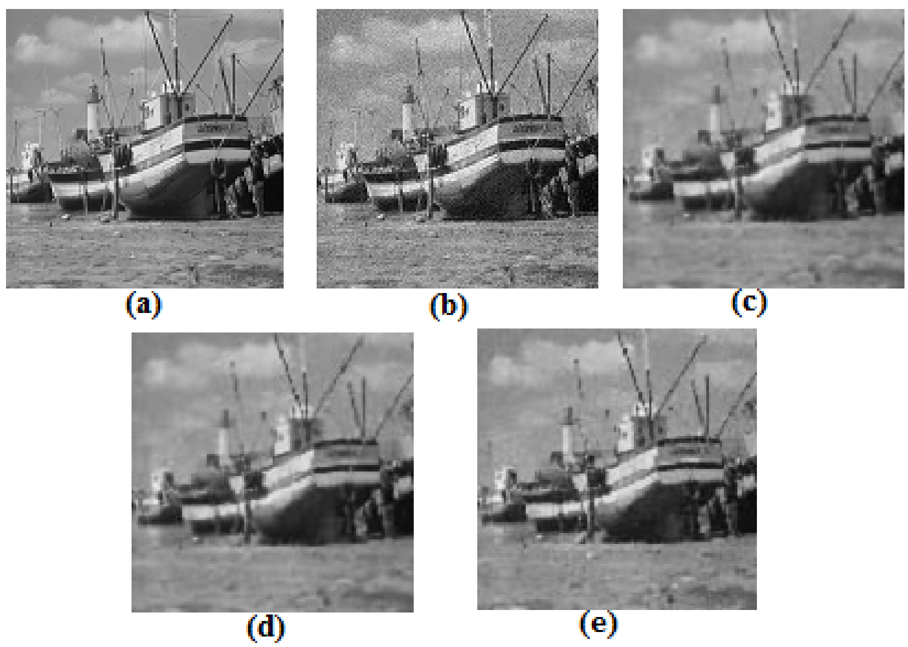

FIGURE 1 – A comparison of results for image restoration : **(a)** original image, **(b)** noised image, **(c)** restored image by the combined model [5], **(d) and (e)** restored image by the proposed model for case 1 and case 2, respectively, $V = 0.01$.

5. REFERENCES

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