APPROXIMATION OF SOLUTIONS FOR RANDOM FRACTIONAL EQUATIONS INVOLVING MEAN SQUARE CAPUTO DERIVATIVES

Hafssa Yfrah[1], *Zoubir Dahmani*[2]

[1,2] Laboratory of Pure and Applied Mathematics,
Faculty of Exact Sciences and Informatics,
University of Mostaganem Abdelhami Bni Badis, 27000, Algeria
hafssa2014yfrah@gmail.com, zzdahmani@yahoo.fr

ABSTRACT

In this research, we investigate the mean square convergence of numerical solutions of the random fractional differential equations using Euler approximation method. The analysis is achieved with the help of the mean square calculus.

1. INTRODUCTION

Fractional differential equations have become a center of interest by many researchers due to their frequent use in different domains. We recommend to the reader [3, 4, 6, 10, 12], and the references therein.
To obtain the analytic solutions of fractional differential equations in some cases is become difficult so establishing approximate mathematical methods for fractional differential equations is important and helpful. There are numerous numerical techniques dedicated towards studying numerical solution for fractional differential equations, such as the Adams predictor-corrector type [1, 2, 11].
As it is known that the parameters of a dynamic system are described as statistics, that is, the information is probabilistic, the common approach in mathematical modeling of such systems is the use of random differential equations or stochastic differential equations. Random fractional differential equations appear in a variety of applications and have been studied by a number of mathematicians [7, 8].
In this study, we are concerned to the following random fractional differential equations with initial conditions :

$$\left\{\begin{array}{rcl} D^{\alpha}U(t) & = & f(t,U(t)), \\ U(0) & = & U_0, \\ U^{'}(0) & = & U_1, \end{array}\right. \tag{1}$$

where, $\alpha \in (1,2]$, $U(\cdot)$ is a random function of order two (... stochastic process) , U_0,U_1 are some random variables of order two, $f : J \times \mathbb{L}_2(\Omega) \to \mathbb{R}$, (with $J = [0,T]$), is given function.
The rest of the article is structured as follow. In section 2, some definitions and lemmas that we need in our study. The construction of the fractional Euler method in section 3, The error analysis for the fractional Euler method are presented in Section 4. And the conclusion is presented in the last section.

2. PRELIMINARIES

Let us start by recalling the following definitions, lemmas and notions [9].

We consider $\mathscr{C} = \mathscr{C}\big(J, \mathbb{L}_2(\Omega)\big)$ the given class of : mean square continuous process of order two, such that

$$\int_J E\big(X^2(t)\big)dt < \infty.$$

The norm is :

$$\|X\|_{\mathscr{C}} = \sup_{t\in J}\|X(t)\|_2 = \sup_{t\in J}\sqrt{E\big(X^2(t)\big)}.$$

Definition 1 *The $\alpha \in (n-1,n]-$ mean square derivative of Caputo for X is :*

$$D^{\alpha}X(t) = I^{n-\alpha}\frac{d^n}{dt}X(t),$$

where, $\frac{d^n}{dt}X(t)$ denotes the mean square differentiation and $X(t)$ is assumed to be mean square differentiable.

Definition 2 *Let us take $\alpha \in (n-1,n]$, where $n \in \mathbf{N}^*$ and $X \in \mathscr{C}$. The mean square integral $I^{\alpha}X(t)$ is :*

$$I^{\alpha}X(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}X(s)ds.$$

Lemma 1 *Let $X(t) \in \mathscr{C}$. For $q > 0$, the general solution of the differential equation $\mathbf{D}^q X(t) = 0$, is given by*

$$X(t) = C_0 + C_1 t - + \cdots + C_{n-1}t^{n-1},$$

where, $C_i \in \mathbb{R}, i = 1,\ldots,n-1, n = [q]+1$.

Lemma 2 *Let $X(t) \in \mathscr{C}$. Let $q > 0$, so,*

$$\mathbf{I}^q\mathbf{D}^q X(t) = X(t) + C_0 + C_1 t + \cdots + C_{n-1}t^{n-1},$$

where, $C_i \in \mathbb{R}, i = 1,\ldots,n-1, n = [q]+1$.

3. MAIN RESULTS

In the following lemma we present the random integral solution of the problem (1)

Lemma 3 *The random integral solution of the problem* (1) *is defined by*

$$U(t) = U_0 + U_1 t + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,U(s))ds, \tag{2}$$

Proof. Applying mean square integral for $\alpha \in]1,2]$ to (1)

$$\mathbf{I}^{\alpha}\mathbf{D}^{\alpha}U(t) = \mathbf{I}^{\alpha}f(t,U(t)),$$

we use the Lemma 2 we get

$$U(t) = U(0) + U^{'}(0)t + \mathbf{I}^{\alpha}f(t,U(t)),$$

using the initial conditions in (1) we get the solution. ■

Now, we pass to present the numerical approximations to the random solution of the problem

(1). We define $0 = t_0 < t_1 < \cdots < t_N = T$, and $t_k = t_0 + kh$, where $h = T/N, N \in \mathbb{N}^*$. Then, an approximation to the integral solution at $t_k, (0 \leq k \leq N)$ can be attained by the following formula

$$\begin{aligned} U(t_k) =& U_0 + U_1 t_k + \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, U(s)) ds, \\ =& U_0 + U_1 t_k + \frac{1}{\Gamma(\alpha)} \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} (t_k - s)^{\alpha-1} f(s, U(s)) ds, t_k \in J, \alpha \in]1,2]. \end{aligned} \tag{3}$$

Using the Euler approximation

$$\int_{t_i}^{t_{i+1}} (t_k - s)^{\alpha-1} f(s, U(s)) ds \approx \int_{t_i}^{t_{i+1}} (t_k - s)^{\alpha-1} f(t_i, U(t_i)) ds, i = 0, \ldots, k-1$$

in (3), we get

$$\begin{aligned} U_k =& U_0 + U_1 t_k + \frac{1}{\Gamma(\alpha)} \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} (t_k - s)^{\alpha-1} f(t_i, U_i) ds, \\ =& U_0 + U_1 t_k + \frac{h^\alpha}{\alpha\Gamma(\alpha)} \sum_{i=0}^{k-1} [(k-i)^\alpha - (k-(i+1))^\alpha] f(t_i, U_i). \end{aligned} \tag{4}$$

4. ERROR ESTIMATES

In this section, we investigate the mean square convergence of numerical schema (4).

Theorem 4 *Assume that the function f satisfies*
$(H1)$: *f is mean square Lipschitz :* $\|f(t,U) - f(t,V)\|_2 \leq L\|U - V\|_2, U, V \in \mathscr{C}$.
$(H2)$: $\left\| \frac{\partial f(t,U)}{\partial t} \right\|_2 \leq l$.
Then the numerical schema given by (4) *is mean square converge.*

Proof. Let e_k denote the error defined as $e_0 = 0$ and $e_k = U_k - U(t_k)$.
We have

$$e_k = \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, U(s)) ds - \frac{h^\alpha}{\alpha\Gamma(\alpha)} \sum_{i=0}^{k-1} [(k-i)^\alpha - (k-(i+1))^\alpha] f(t_i, U_i).$$

Therefore

$$\begin{aligned} \left\| e_k \right\|_2 \leq& \left\| \int_0^{t_k} \frac{(t_k - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, U(s)) ds - \frac{h^\alpha}{\alpha\Gamma(\alpha)} \sum_{i=0}^{k-1} [(k-i)^\alpha - (k-(i+1))^\alpha] f(t_i, U(t_i)) \right\|_2 \\ &+ \frac{h^\alpha}{\alpha\Gamma(\alpha)} \sum_{i=0}^{k-1} [(k-i)^\alpha - (k-(i+1))^\alpha] \left\| f(t_i, U(t_i)) - f(t_i, U_i) \right\|_2. \end{aligned}$$

We have

$$\begin{aligned}
&\left\| \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,U(s))ds - \frac{h^\alpha}{\alpha\Gamma(\alpha)} \sum_{i=0}^{k-1} [(k-i)^\alpha - (k-(i+1))^\alpha] f(t_i,U(t_i)) \right\|_2 \\
&= \left\| \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} (t_k-s)^{\alpha-1} [f(t,U(t)) - f(t_i,U(t_i))] ds \right\|_2 \\
&\leq \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} (t_k-s)^{\alpha-1} [\|f(t,U(t)) - f(t,U(t_i))\|_2 + \|f(t,U(t_i)) - f(t_i,U(t_i))\|_2] ds \\
&\leq \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} (t_k-s)^{\alpha-1} [L(U(t)-U(t_i)) + (t-t_i)l] ds
\end{aligned}$$

We have $U(t)-U(t_i) = \int_{t_i}^{t} U'(s)ds \leq h \max_{s\in J} U'(s)$.
By simple calculation, we get

$$\begin{aligned}
&\left\| \int_0^{t_k} \frac{(t_k-s)^{\alpha-1}}{\Gamma(\alpha)} f(s,U(s))ds - \frac{h^\alpha}{\alpha\Gamma(\alpha)} \sum_{i=0}^{k-1} [(k-i)^\alpha - (k-(i+1))^\alpha] f(t_i,U(t_i)) \right\|_2 \\
&\leq \frac{t_k^\alpha}{\alpha} \omega h,
\end{aligned}$$

where $\omega = L \max_{s\in J} U'(s) + l$.
In the other hand, we have

$$\begin{aligned}
&\frac{h^\alpha}{\alpha\Gamma(\alpha)} \sum_{i=0}^{k-1} [(k-i)^\alpha - (k-(i+1))^\alpha] \|f(t_i,U(t_i)) - f(t_i,U_i)\|_2, \\
&\leq \frac{Lh^\alpha}{\alpha\Gamma(\alpha)} \sum_{i=0}^{k-1} [(k-i)^\alpha - (k-(i+1))^\alpha] \|e_i\|_2, \\
&\leq \frac{Lh^\alpha}{\Gamma(\alpha)} \sum_{i=0}^{k-1} [(k-i)^{\alpha-1}] \|e_i\|_2 \\
&\leq \frac{LhT^{\alpha-1}}{\Gamma(\alpha)} \sum_{i=0}^{k-1} \|e_i\|_2
\end{aligned}$$

At the end, we get

$$\|e_k\|_2 \leq \frac{t_k^\alpha}{\alpha} \omega h + \frac{LhT^{\alpha-1}}{\Gamma(\alpha)} \sum_{i=0}^{k-1} \|e_i\|_2.$$

Applying the Gronwall inequality [5][p.860, Lemma 3.5] we find

$$\|e_k\|_2 \leq \zeta h, \quad \zeta \text{is a constant.}$$

When h is small enough the numerical schema (4) is converge in the sense of mean square. The prove is complete. ■

5. CONCLUSIONS

The topic of this research is to approximate the random solution of the fractional differential equations using fractional Euler method. We demonstrate that the proposed method is mean square convergent. For upcoming work, we will study other types of numerical schema for random fractional differential equations.

6. REFERENCES

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