UNIFORM CONVERGENCE OF NONPARAMETRIC CONDITIONAL HAZARD FUNCTION IN THE SINGLE FUNCTIONAL MODELING FOR DEPENDENT DATA

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ABSTRACT

We study the nonparametric local linear estimation of the conditional hazard function of a scalar response variable given a functional explanatory variable, when the functional data are α-mixing dependency and we give the uniform almost complete convergence with rates of this function.

1. INTRODUCTION

The contribution of this work is to study the conditional hazard in the single functional index model, for its excellence in many characteristics and due to the flexibility of the model in dimension reduction and used in econometrics fields as accord between nonparametric and parametric models. The single-index models have been considered in the multivaraite case by Hardle et al. (1993), Hristache et al. (2001) and Delecroix et al. (2003). Then, by nonparametric kernel estimation, Ferraty et al. (2003) started to deal with the single functioanl index, they obtained the almost complete convergence in the independent and identically distributed (i.i.d) case for regression function. Particularly, in the quasi-associated, Hadjila and Ait Saidi (2018) studied the pointwise almost complete convergence and the uniform almost complete convergence(with the rate) of the kernel estimate of the hazard function of a real random variable conditioned by a functional predictor, also, gave a simulation to illustrate their methodology.
In our study, we estimate the conditional hazard in the single index model for dependent data of a real variable Y given a functional variable X in the local linear method (see, Barrientos et al. (2010)). We point out that the single functional index in this method is intimately limited until now. Our work count on the study of the conditional hazard function of a scalar response variable Y given a Hilbertian random variable in functioanl single-index model for dependence case in the local linear method, such that under certain conditions we prove its uniform almost complete convergence.
In this paper, we will see the model and the estimator in the local linear estimation in section 2. Then, we give in section 3 assumptions and results. Finally, we finished by a conclusion.

2. MODEL

Let $\{Y_i, X_i\}_{i\in\mathbb{N}}$ be a random processes identically distributed as (Y, X) where Y_i's are valued in $\mathbb{R}$ and X_i takes values in seperable Hilbert space $\mathscr{H}$ with the norm $||.||$ generated by an inner product $< .,. >$. We assume that the regular version of the conditional probability of Y given X exists and bounded. Moreover, we suppose that the conditional hazard function of Y given X has a known single-index θ in $\mathscr{H}$ and we denote the conditional density by $f^x_\theta(y)$ respect to

Lebesgue's measure over $\mathbb{R}$. So, denote the conditional hazard function of Y given X by

$$h_\theta^x(y) = \frac{f_\theta^x(y)}{1 - F_\theta^x(y)}, \ \ \forall y \in \mathbb{R}$$

where, $F_\theta^x(y) < 1$.
As usually, in the single-index model the identifiability is assured such that, $\forall x \in \mathscr{H}$, we have

$$h_1(y| <\cdot, \theta_1>) = h_2(y| <\cdot, \theta_2>) \Rightarrow h_1 \equiv h_2 \text{ and } \theta_1 = \theta_2.$$

The local linear estimator (see, Demongeot al. (2010)) of $F_\theta^x(y)$ and $f_\theta^x(y)$ was defined as follows

$$\widehat{F}_\theta^x(y) = \frac{\sum\limits_{1 \le i,j \le n} W_{\theta,ij}(x) H({h_H}^{-1}(y - Y_j))}{\sum\limits_{1 \le i,j \le n} W_{\theta,ij}(x)}$$

and

$$\widehat{f}_\theta^x(y) = \frac{\sum\limits_{1 \le i,j \le n} W_{\theta,ij}(x) H'({h_H}^{-1}(y - Y_j))}{h_H \sum\limits_{1 \le i,j \le n} W_{\theta,ij}(x)},$$

with

$$W_{\theta,ij}(x) = \beta_\theta(X_i,x)\Big(\beta_\theta(X_i,x) - \beta_\theta(X_j,x)\Big) K(h_K^{-1} d_\theta(x,X_i)) K(h_K^{-1} d_\theta(x,X_j))$$

with
$\beta_\theta(X_i,x) = <x - X_i, \theta>$ is a known bi-functional operator from $\mathscr{H}^2$ into $\mathbb{R}$, such that $\forall x_1, x_2 \in \mathscr{H}$, $\forall \theta \in \mathscr{H}$, d_θ is a semi-metric associated to the single index $\theta \in \mathscr{H}$ defined by $d_\theta(x_1,x_2) := |<x_1 - x_2, \theta>|$, with the kernel K. H is a distribution function (respectively, H' is the derivative of H), and $h_K = h_{K,n}$(respectively, $h_H = h_{H,n}$) is a sequence of positive real numbers.
Finally, the local linear estimator of the hazard function is given by

$$\widehat{h}_\theta^x(y) = \frac{\widehat{f}_\theta^x(y)}{1 - \widehat{F}_\theta^x(y)}.$$

Now, we define the definition of $\alpha-$mixing sequence. The sequence is said to be $\alpha-$mixing (strong mixing), if the mixing coefficient $\alpha(n) \xrightarrow{n \to \infty} 0$ such that

$$\alpha(n) = \sup_k \sup_{A \in \sigma_1^k(X), B \in \sigma_{n+k}^\infty(X)} \{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|, k \in \mathbb{N}^*\}$$

and σ_j^k denote the σ-algebra generated by the random variales $\{(Y_i,X_i), j \le i \le k\}$.

3. ASSUMPTIONS AND RESULTS

3.1. Uniform almost complete convergence

In this paper, we will study the uniform almost complete convergence denote by C, C' and C'', also, $C_{\theta,x}$, some strictly positive constants, and $\forall x \in \mathscr{H}$, and $i, j = 1, ., n$,

$$K_{\theta,i}(x) := K(h_K^{-1} d_\theta(x,X_i)) \text{ and }, \ \forall y \in \mathbb{R}, \ H_j(y) := H({h_H}^{-1}(y - Y_j)).$$

On the other hand, we denotes x a fixed point in $\mathscr{H}$, $\mathscr{N}_x$ is a fixed neighborhood of x and $S_{\mathbb{R}}$ is a fixed compact of $\mathbb{R}$. We consider the following cover of the compacts $S_{\mathscr{H}}$ and $\Theta_{\mathscr{H}}$:

$$S_{\mathscr{H}} \in \bigcup_{j=1}^{N^{S_{\mathscr{H}}}} B(x_j, r_n) \text{ and } \Theta_{\mathscr{H}} \in \bigcup_{j''=1}^{N^{\Theta_{\mathscr{H}}}} B(t_{j''}, r_n)$$

and $\forall x \in \mathscr{H}, \forall \theta \in \Theta_{\mathscr{H}}$ we set

$$j(x) = \arg \min_{j \in \{1,\dots,N^{S_{\mathscr{H}}}\}} ||x - x_j|| \text{ and } j''(\theta) = \arg \min_{j'' \in \{1,\dots,N^{\Theta_{\mathscr{H}}}\}} ||\theta - t''_j||$$

where, $x_j, t_{j''} \in \mathscr{H}$ and r_n is a sequence of positif numbers.
Suppose that $N^{S_{\mathscr{H}}}, N^{\Theta_{\mathscr{H}}}$, are the minimal numbers of open balls (see, Kolmogorov et Tikhomirov (1959)) with radius r_n in $\mathscr{H}$, which are required to cover $S_{\mathscr{H}}$ and $\Theta_{\mathscr{H}}$.

For our context, we need assumptions for our estimate.

$$\forall h_K > 0,\ \mathbb{P}(| < X - x, \theta > | < h_K) =: \phi_{\theta,x}(h_K) > 0.$$

(U1) There exists a differentiable function $\phi(\cdot)$ such that $\forall x \in S_{\mathscr{H}}$, and $\forall \theta \in \Theta_{\mathscr{H}}$

$$0 < C\phi(h_K) \le \phi_{\theta,x}(h_K) \le C\phi'(h_K) < \infty \quad \text{and } \exists \eta_0 > 0,\ \forall \eta < \eta_0, \phi'(\eta) < 0,$$

where ϕ' is the first derivative function of ϕ, and $\phi(0) = 0$.

(U2) The function F^x_θ and f^x_θ satisfy :

$$\begin{cases} \exists a_1, a_2 > 0,\ \forall (y,y') \in S^2_{\mathbb{R}}, \forall (x,x') \in \mathscr{N}_x \times \mathscr{N}_x,\ \forall \theta \in \Theta, \\ (i)\ |f^x_\theta(y) - f^{x'}_\theta(y')| \le C'(||x-x'||^{a_1} + |y-y'|^{a_2}), \\ (ii)\ |F^x_\theta(y) - F^{x'}_\theta(y')| \le C''(||x-x'||^{a_1} + |y-y'|^{a_2}). \end{cases}$$

(U3) The pairs $(X_i, Y_j)_{i,j \in \mathbb{N}}$ satisfies :
(i) $\exists a > 0, \exists c > 0 : \forall n \in \mathbb{N}, \alpha(n) \le cn^{-a}$.
(ii) $\exists 0 < d \le 1, 0 \le C\phi(h_K)^{1+d} \le \varphi_{\theta,x}(h_K) \le C'\phi(h_K)^{1+d}$.
where $\varphi_{\theta,x}(h_K) := \sup_{i \ne j} \mathbb{P}\Big((X_i, X_j) \in B(x,h_K) \times B(x,h_K)\Big)$.

(U4) The bi-functional function $\beta_\theta(\cdot,\cdot)$ is Lipschitsian continuos function and satisfying :

$$\forall x' \in S_{\mathscr{H}},\ Cd_\theta(x',x) \le |\beta_\theta(x,x')| \le C'd_\theta(x',x).$$

(U5) (i) The kernel K is a positive, Lipschitzian and differentiable function, supported within $(-1,1)$.
(ii) The kernel H is a positive, bounded and Lipschitzian continuous function, such that :

$$\int |t|^{a_2} H(t) \mathrm{d}t < \infty \text{ and } \int H^2(t) \mathrm{d}t < \infty.$$

(U6) The bandwidth h_K satisfies : $\exists n_0 \in \mathbb{N},\ \forall \eta > n_0,\ -\frac{1}{\phi_{\theta,x}(h_K)} \int_{-1}^1 \phi_{\theta,x}(th_K, h_K) \frac{\mathrm{d}}{\mathrm{d}t}(t^2 K(t)) \mathrm{d}t > C'' > 0$ and $h_K \int_{B(x,h_K)} \beta_\theta(u,x) \mathrm{d}P(u) = o\Big(\int_{B(x,h_K)} \beta^2_\theta(u,x) \mathrm{d}P(u)\Big)$
where $B(x,h) = \{z \in \mathscr{H} |\ d_\theta(z,x) \le h\}$ and $\mathrm{d}P(u)$ is the probability measure of X.

(U7) For some $\lambda > 0$ the bandwidth h_H satisfies

$$\lim_{n\to\infty} n^\lambda h_H = \infty, \text{ and } \lim_{n\to\infty} \frac{\ln n}{n h^{(j)}_H \phi(h_K)} = 0, \text{ for } j = 0,1.$$

(U8) $\exists 0 < t < 1,\ Cn^{\frac{3-a}{a+1}+\eta_0} \le h^{(j)}_H \phi(h_K) \le Cn^{-t}$
where $\eta_0 > \frac{\lambda+1}{a+1}$, for $j = 0,1$.

Comments on assumption

As usually in functional statistics and in the independent case, the conditions (U1) and (U4) are standard hypotheses (see, Ferraty et al. (2003)). (U2) is about regularity and boundary conditions. Hypotheses (U5) and (U7) are a technical conditions (see, Barrientos et al. (2010)). Particulary, for the dependence frame, (U3) indicate that the observations are α-mixing dependency. Likewise, we find the condition (U6) in Barrientos et al. (2010) and we put (U8) that needed for our asymptotic results.

Theorem 1 *Under assumptions* $(U1)-(U8)$*, we have :*

$$\sup_{\theta\in\Theta_{\mathscr{H}}}\sup_{x\in S_{\mathscr{H}}}\sup_{y\in S_{\mathbb{R}}}|\widehat{h}^x_\theta(y)-h^x_\theta(y)| = O(h_K^{a_1}+h_H^{a_2})+O_{a.co.}\left(\sqrt{\frac{\ln(N^{S_{\mathscr{H}}}N^{\Theta_{\mathscr{H}}})}{nh_H\phi(h_K)}}\right).$$

Proof of Theorem 1. The proof is based on the decomposition in Theorem 3.1 of Merouan *et al.* (2019) which we remind it and the Lemmas below.

$$\widehat{h}^x_\theta(y)-h^x_\theta(y)=\frac{1}{1-\widehat{F}^x_\theta(y)}\left(\widehat{f}^x_\theta(y)-f^x_\theta(y)\right)+\frac{h^x_\theta(y)}{1-\widehat{F}^x_\theta(y)}\left(\widehat{F}^x_\theta(y)-F^x_\theta(y)\right). \tag{1}$$

Where for $p=0,1$, we have :

$\widehat{F}_\theta^{x^{(p)}}(y)-F_\theta^{x^{(p)}}(y)=\frac{1}{\widehat{g}^x_{\theta,D}}\left\{\left(\widehat{F}^{x^{(p)}}_{\theta,N}(y)-\mathbb{E}[\widehat{F}^{x^{(p)}}_{\theta,N}(y)]\right)-\left(F^{x^{(p)}}_\theta(y)-\mathbb{E}[\widehat{F}^{x^{(p)}}_{\theta,N}(y)]\right)\right\}+\frac{F^{x^{(p)}}_\theta(y)}{\widehat{g}^x_{\theta,D}}(1-\widehat{g}^x_{\theta,D})$

and

$\widehat{F}^{x^{(p)}}_{\theta,N}(y)=\frac{1}{n(n-1)h_H^{(p)}\mathbb{E}[W_{\theta,12}(x)]}\sum_{1\le i\ne j\le n}W_{\theta,ij}(x)H^{(p)}(h_H^{-1}(y-Y_j))$

$\widehat{g}^x_{\theta,D}=\frac{1}{n(n-1)\mathbb{E}[W_{\theta,12}(x)]}\sum_{1\le i\ne j\le n}W_{\theta,ij}(x).$

Under assumptions $(U1),(U2)$ and $(U5)$, we obtain :

$$\sup_{\theta\in\Theta_{\mathscr{H}}}\sup_{x\in S_{\mathscr{H}}}\sup_{y\in S_{\mathbb{R}}}|f^x_\theta(y)-\mathbb{E}[\widehat{f}^x_{\theta,N}(y)]|=O(h_K^{a_1})+O(h_H^{a_2}).$$

and

$$\sup_{\theta\in\Theta_{\mathscr{H}}}\sup_{x\in S_{\mathscr{H}}}\sup_{y\in S_{\mathbb{R}}}|F^x_\theta(y)-\mathbb{E}[\widehat{F}^x_{\theta,N}(y)]|=O(h_K^{a_1})+O(h_H^{a_2}).$$

Under assumptions $(U1)$ and $(U3)-(U8)$, we get :

$$\sup_{\theta\in\Theta_{\mathscr{H}}}\sup_{x\in S_{\mathscr{H}}}|1-\widehat{g}^x_{\theta,D}|=O_{\text{a.co.}}\left(\sqrt{\frac{\ln(N^{S_{\mathscr{H}}}N^{\Theta_{\mathscr{H}}})}{n\phi(h_K)}}\right)\text{ and }\sum_{i=1}^{\infty}\mathbb{P}(\inf_{\theta\in\Theta_{\mathscr{H}}}\inf_{x\in S_{\mathscr{H}}}\widehat{g}^x_{\theta,D}<1/2)<\infty.$$

Under assumptions $(U1),(U2)-(U8)$, we obtain :

$$\sup_{\theta\in\Theta_{\mathscr{H}}}\sup_{x\in S_{\mathscr{H}}}\sup_{y\in S_{\mathbb{R}}}|\widehat{f}^x_{\theta,N}(y)-\mathbb{E}[\widehat{f}^x_{\theta,N}(y)]|=O_{\text{a.co.}}\left(\sqrt{\frac{\ln(N^{S_{\mathscr{H}}}N^{\Theta_{\mathscr{H}}})}{nh_H\phi(h_K)}}\right).$$

and

$$\sup_{\theta\in\Theta_{\mathscr{H}}}\sup_{x\in S_{\mathscr{H}}}\sup_{y\in S_{\mathbb{R}}}|\widehat{F}^x_{\theta,N}(y)-\mathbb{E}[\widehat{F}^x_{\theta,N}(y)]|=O_{\text{a.co.}}\left(\sqrt{\frac{\ln(N^{S_{\mathscr{H}}}N^{\Theta_{\mathscr{H}}})}{n\phi(h_K)}}\right).$$

Corollary 2 *Under the conditions of Theorem* **??***, we get :*

$$\exists\mu>0,\ \sum_{n=1}^{\infty}\mathbb{P}\left(\inf_{\theta\in\Theta_{\mathscr{H}}}\inf_{x\in S_{\mathscr{H}}}\inf_{y\in S_{\mathbb{R}}}|1-\widehat{F}^x_\theta(y)|<\mu\right)<\infty.$$

4. CONCLUSION

In this paper, we establish the estimation of the conditional hazard function in the single functional index model for α-mixing functional data. Under some conditions, we present the uniform almost complete convergence of the local linear estimator with rate.

5. REFERENCES

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