# ON THE STRONGLY MID P-SUMMING OPERATORS AND APPLICATION

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**ABSTRACT**

In this paper, we introduce and study the new ideal of strongly mid $p$-summing linear operators between Banach spaces. We prove, an operator is strongly mid $p$-summing if and only if its adjoint is absolutely mid $p^*$-summing. This result led us to prove, $X$ is $p$-Dunford-Pettis property $(1 < p < \infty)$ if and only if, so is $X^*$.

## 1. NOTATION AND PRELIMINARIES

The notation used in the paper is in general standard. The letters $X, X_1, \cdots, X_m, Y$ ($m$ be in $\mathbb{N}$ ) shall denote Banach spaces over $\mathbb{K}$ (real or complex scalars field). We will denote by $\mathscr{L}(X_1, \cdots, X_m; Y)$ the Banach space of all bounded multilinear operators from $X_1 \times \cdots \times X_m$ into $Y$ equipped with the operator norm. $\mathscr{L}(X;Y)$ the Banach space of all bounded linear operators $T : X \to Y$ endowed with the usual sup norm. The closed unit ball of $X$ is denoted by $B_X$ and its topological dual by $X^* = \mathscr{L}(X;\mathbb{K})$.

For a Banach space $X$, and $1 \leq p \leq \infty$, let $p^*$ be its conjugate that is $1/p + 1/p^* = 1$. Let us define the sequences spaces we shall work with : (see [3, 2]).
$\ell_p(X) =$ the Banach space of absolutely $p$-summable sequences with the norm $(1 \leq p < \infty)$

$$\| \left(x_j\right)_{j=1}^{\infty} \|_p := \left( \sum_{j=1}^{\infty} \| x_j \|^p \right)^{1/p}.$$

$\ell_p^w(X) =$ the Banach space of weakly $p$-summable sequences with the norm $(1 \leq p < \infty)$

$$\| (x_j)_{j=1}^{\infty} \|_{w,p} := \sup_{x^* \in B_{X^*}} \left( \sum_{j=1}^{\infty} \left| x^* \left(x_j\right) \right|^p \right)^{1/p}.$$

$\ell_p^{mid}(X)$ the Banach space of mid $p$-summable sequences with the norm $(1 \leq p < \infty)$

$$\| \left(x_j\right)_{j=1}^{\infty} \|_{mid,p} := \sup_{(x_n^*)_{n=1}^{\infty} \in B_{\ell_p^w(X^*)}} \left( \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \left| x_n^* \left(x_j\right) \right|^p \right)^{1/p}.$$

When $p = \infty$ we have $\| \left(x_j\right)_{j=1}^{\infty} \|_\infty = \| \left(x_j\right)_{j=1}^{\infty} \|_{w,\infty} = \| \left(x_j\right)_{j=1}^{\infty} \|_{mid,\infty} = \sup_j \|x_j\|$. The relationships between the various sequence spaces are given by [2]

$$\ell_p(X) \subsetneq \ell_p^{mid}(X) \subsetneq \ell_p^w(X),$$

with

$$\| \left(x_j\right)_{j=1}^{\infty} \|_p \leq \| \left(x_j\right)_{j=1}^{\infty} \|_{w,p} \leq \| \left(x_j\right)_{j=1}^{\infty} \|_{mid,p}.$$

Recall that, for $1 \leq p < \infty$, we say that a continuous linear operator $T : X \to Y$ is absolutely $p$-summing if

$$(T(x_j))_{j=1}^{\infty} \in \ell_p(Y) \textit{ whenever } (x_j)_{j=1}^{\infty} \in \ell_p^w(Y).$$

The class of absolutely $p$-summing operator from $X$ to $Y$ will be represented by $\Pi_p(X;Y)$. An equivalent formulation asserts that $T$ is absolutely $p$-summing if there is a positive constant $C$ such that

$$\|(T(x_j))_{j=1}^{\infty}\|_p \leq C\|(x_j)_{j=1}^{\infty}\|_{w,p}$$

for all $(x_j)_{j=1}^{\infty} \in \ell_p^w(X)$. The infimum of all $C > 0$ that satisfy the above inequality define a norm, denoted by $\pi_p(T)$ .

Let $(\mathscr{I}(X;Y), \alpha(\cdot))$ be an operator ideal, we put

$$\mathscr{I}^d(X;Y) = \{T \in \mathscr{L}(X;Y) : T^* \in \mathscr{I}(Y^*;X^*)\}.$$

For an operator $T \in \mathscr{I}^d(X;Y)$, we put $\alpha^d(T) = \alpha(T^*)$. With these notations $(\mathscr{I}^d(X;Y), \alpha^d(\cdot))$ is also a Banach operator ideal and is called the dual ideal of $(\mathscr{I}(X;Y), \alpha(\cdot))$ (see [6, Section 4]).

Let us recall now the definition of strongly $p$-summing linear operators. An operator $T \in \mathscr{L}(X;Y)$ is strongly $p$-summing if there is a constant $C$ such that

$$\|(\langle T(x_j), y_j^*\rangle)_{j=1}^{\infty}\|_1 \leq C\|(x_j)_{j=1}^{\infty}\|_p \|(y_j^*)_{j=1}^{\infty}\|_{w,p^*}$$

for all $(x_j)_{j=1}^{\infty} \in \ell_p(X)$ and $(y_j^*)_{j=1}^{\infty} \in \ell_{p^*}^w(Y^*)$. The space $\mathscr{D}_p(X;Y)$ of all strongly $p$-summing linear operators from $X$ into $Y$ which is a Banach space with the norm $d_p(T)$ the infimum of all $C > 0$ that satisfy the above inequality. According to [5, 7] we obtain $\mathscr{D}_p^d(X;Y) = \Pi_{p^*}(X;Y)$ and $\Pi_p^d(X;Y) = \mathscr{D}_{p^*}(X;Y)$.

Our results are presented as follows. In first section 1, we recall important results and definitions to be used later. In section 2, we introduce and investigate a new ideal of strongly mid $p$-summing operators. We present a characterization given by a summability property. We also prove the related dual result : an operator $T \in \mathscr{L}(X;Y)$ is strongly mid $p$-summing ($T \in \mathscr{D}_p^{mid}(X;Y)$) if and only if its adjoint $T^*$ is absolutely mid $p^*$-summing ($T^* \in \Pi_{p^*}^{mid}(Y^*;X^*)$). We study Banach space $X$ for which $id_X \in \Pi_p^{mid}$(Theorem 6), $id_X \in \mathscr{D}_p^{mid}$(Theorem 8), $\mathscr{D}_p(\ell_p;X) = \mathscr{L}(\ell_p;X)$ (Theorem 10). We also prove, $X$ has the $p$-Dunford-Pettis property if and only if $X^*$ has the $p$-Dunford-Pettis property.

## 2. STRONGLY MID SUMMING LINEAR OPERATORS

Botelho, Campos and Santos [2] introduced the concept of absolutely mid $p$-summing operators. For $1 \leq p < \infty$, an operator $T \in \mathscr{L}(X;Y)$ is absolutely mid $p$-summing if

$$(T(x_j))_{j=1}^{\infty} \in \ell_p^{mid}(Y) \textit{ whenever } (x_j)_{j=1}^{\infty} \in \ell_p(X).$$

By $\Pi_p^{mid}(X;Y)$, we denote the space of absolutely mid $p$-summing operators.

We introduce the concept of strongly mid $p$-summing linear operators as a characterization of the conjugates of absolutely mid $p^*$-summing linear operators. This idea seems to have appeared for the first time in [5] and [7].

**Definition 1** *Let $1 < p \leq \infty$. A mapping $T \in \mathscr{L}(X;Y)$ is strongly mid p-summing if there exist a constant $C > 0$ such that*

$$\|(\langle T(x_j), y_j^*\rangle)_{j=1}^{k}\|_1 \leq C\|(x_j)_{j=1}^{k}\|_p \|(y_j^*)_{j=1}^{k}\|_{mid,p^*} \tag{1}$$

*for any $(x_j)_{j=1}^{k} \subset X$ and $(y_j^*)_{j=1}^{k} \subset Y^*$. $\mathscr{D}_p^{mid}(X;Y)$ denotes the space of all strongly mid p-summing operators from $X$ to $Y$. The least $C$ for which (1) holds will be written $d_p^{mid}(T)$. From the definition it is clear that $\mathscr{D}_p(X;Y) \subset \mathscr{D}_p^{mid}(X;Y)$ and $d_p^{mid}(\cdot) \leq d_p(\cdot)$.*

For $T \in \mathscr{L}(X;Y)$. The induced map $\varphi_T : X \times Y^* \to \mathbb{K}$ given by $\varphi_T(x,y^*) = \langle T(x), y^* \rangle$ for all $x \in X$ and $y^* \in Y^*$ is continuous bilinear. Using the abstract approach of [1] and Proposition 1.9 in [2], we can see that the next proposition is immediate consequences of [1, Proposition 2.4].

**Proposition 1** *Let $T \in \mathscr{L}(X;Y)$, the following assertions are equivalent.*

1. $T \in \mathscr{D}_p^{mid}(X;Y)$.
2. $(\varphi_T(x_j, y_j^*))_{j=1}^{\infty} \in \ell_1$ *whenever* $(x_j)_{j=1}^{\infty} \in \ell_p(X)$ *and* $(y_j^*)_{j=1}^{\infty} \in \ell_{p^*}^{mid}(Y^*)$.
3. *The induced map* $\widehat{\varphi_T} : \ell_p(X) \times \ell_{p^*}^{mid}(Y^*) \to \ell_1$ *given by* $\widehat{\varphi_T}((x_j)_{j=1}^{\infty}, (y_j^*)_{j=1}^{\infty}) = (\varphi_T(x_j, y_j^*))_{j=1}^{\infty}$ *is a well-define and continuous bilinear operator.*
4. *There exist a constant $C > 0$ such that*
$$\|(\varphi_T(x_j, y_j^*))_{j=1}^{\infty}\|_1 \leq C \|(x_j)_{j=1}^{\infty}\|_p \|(y_j^*)_{j=1}^{\infty}\|_{mid,p^*}. \tag{2}$$
*for any* $(x_j)_{j=1}^{\infty} \in \ell_p(X)$ *and* $(y_j^*)_{j=1}^{\infty} \in \ell_{p^*}^{mid}(Y^*)$.
5. *There exist a constant $C > 0$ such that*
$$\|(\varphi_T(x_j, y_j^*))_{j=1}^{k}\|_1 \leq C \|(x_j)_{j=1}^{k}\|_p \|(y_j^*)_{j=1}^{k}\|_{mid,p^*}. \tag{3}$$
*for any* $(x_j)_{j=1}^{k} \subset X$ *and* $(y_j^*)_{j=1}^{k} \subset Y^*$.

*Moreover,* $d_p^{mid}(T) = \|\widehat{\varphi_T}\| = \inf\{C : (2)\ holds\} = \inf\{C : (3)\ holds\}$.

**Theorem 2** $(\mathscr{D}_p^{mid}(X;Y), d_p^{mid}(\cdot))$ *is a Banach operator ideal.*

**Proof.** Using the abstract framework, notation and language [1], we find that a linear operator $T$ is strongly $p$-summing if and only if $\varphi_T$ is $(\ell_p(.)\ell_{p^*}^{mid}(.);\ell_1)$-summing. Since $1/p + 1/p^* = 1$ we obtain

$$\|(\lambda_j^1.\lambda_j^2)_{j=1}^{\infty}\|_1 \leq \|(\lambda_j^1)_{j=1}^{\infty}\|_{\ell_p(\mathbb{K})} \|(\lambda_j^2)_{j=1}^{\infty}\|_{\ell_{p^*}(\mathbb{K})} = \|(\lambda_j^1)_{j=1}^{\infty}\|_{\ell_p(\mathbb{K})} \|(\lambda_j^2)_{j=1}^{\infty}\|_{\ell_{p^*}^{mid}(\mathbb{K})}.$$

Therefor, $\ell_p(\mathbb{K})\ell_{p^*}^{mid}(\mathbb{K}) = \ell_p(\mathbb{K})\ell_{p^*}(\mathbb{K}) \overset{1}{\hookrightarrow} \ell_1$. In addition, all the sequence classes involved are linearly stable (see [2, Proposition 1.10]). So, form [1, Theorem 3.6] it follows that $(\mathscr{D}_p^{mid}, d_p^{mid}(.))$ is a Banach operators ideal. ■ In the next theorem, we will show that in fact the absolutely mid $p$-summing linear operators is the adjoint of strongly mid $p$-summing linear operators that will be useful throughout our section.

**Theorem 3** *Let $1 < p \leq \infty$, $T \in \mathscr{L}(X;Y)$. Then $T \in \mathscr{D}_p^{mid}(X;Y)$ if and only if $T^* \in \Pi_{p^*}^{mid}(Y^*;X^*)$. Moreover, $d_p^{mid}(T) = \pi_{p^*}^{mid}(T^*)$.*

**Proof.** Assume that $T \in \mathscr{D}_p^{mid}(X;Y)$. For $(x_j)_{j=1}^{\infty} \in \ell_p(X)$ and $(y_j^*)_{j=1}^{\infty} \in \ell_{p^*}^{mid}(Y^*)$. We have

$$\|(\langle T(x_j), y_j^* \rangle)_{j=1}^{\infty}\|_1 = \|(\langle T^*(y_j^*), x_j \rangle)_{j=1}^{\infty}\|_1 \leq d_p^{mid}(T) \|(x_j)_{j=1}^{\infty}\|_p \|(y_j^*)_{j=1}^{\infty}\|_{mid,p^*}$$

by taking the supremum over the unit ball in $\ell_p(X)$ we obtain $\|(T^*(y_j^*))_{j=1}^{\infty}\|_{p^*} \leq C \|(y_j^*)_{j=1}^{\infty}\|_{mid,p^*}$. Therefore, $T^* : Y^* \to X^*$ is absolutely mid $p^*$-summing and $d_p^{mid}(T) \geq \pi_p^{mid}(T^*)$. Conversely, let $T^* \in \Pi_{p^*}^{mid}(Y^*;X^*)$. For $(x_j)_{j=1}^{\infty} \in \ell_p(X)$ and $(y_j^*)_{j=1}^{\infty} \in \ell_{p^*}^{mid}(Y^*)$. By Hölder's inequality we have

$$\sum_{j=1}^{\infty} |\langle T(x_j), y_j^* \rangle| \leq \|(x_j)_{j=1}^{\infty}\|_p \|(T^*(y_j^*))_{j=1}^{\infty}\|_{p^*} \leq \pi_p^{mid}(T^*) \|(x_j)_j^{\infty}\|_p \|(y_j^*))_{j=1}^{\infty}\|_{mid,p^*}.$$

Therefore $T$ is strongly mid $p$-summing and $d_p^{mid}(T) = \pi_p^{mid}(T^*)$. ■ As a consequence, we obtain the following corollary which is a straightforward consequence of the preceding theorem and Theorem 2.8 in [2].

**Corollary 4** *Let* $1 \leq p < \infty$, $T \in \mathscr{L}(X;Y)$. *If* $T^* \in \mathscr{D}_{p^*}^{mid}(Y^*;X^*)$ *then* $T \in \Pi_p^{mid}(X;Y)$.

Recall that an operator ideal $\mathscr{I}(X;Y)$ is surjective if $T \in \mathscr{I}(X;Y)$ whenever $T \circ Q \in \mathscr{I}(X_0;Y)$, where $Q : X_0 \to X$ is the quotient map. As consequence, we obtain the following proposition which is a straightforward consequence of Theorem 3, and [2, Proposition 2.8] and [6, Section 4].

**Proposition 5** *The operator ideal* $(\mathscr{D}_p^{mid}, d_p^{mid})$ *is surjective.*

The Dvoretzky–Rogers theorem (see [3]) states that the Banach space $X$ is finite dimensional if, and only if, $id_X$ is $p$-summing. Now we will give a results in this context.

We say that a Banach space $X$ is a weak mid $p$-space (or $p$-Dunford-Pettis property see [4]) if $\ell_p^{mid}(X) = \ell_p^w(X)$ and it is a strong mid $p$-space if $\ell_p^{mid}(X) = \ell_p(X)$.

The next result is a reformulation of [2, Theorem 2.7].

**Theorem 6** *The following are equivalent :*
*1)* $X$ *is a strong mid* $p$*-space.*
*2)* $id_X \in \Pi_p^{mid}(X;X)$.
*3)* $\mathscr{L}(X;Y) = \Pi_p^{mid}(X;Y)$ *for every Banach space* $Y$.
*4)* $\mathscr{L}(Y;X) = \Pi_p^{mid}(Y;X)$ *for every Banach space* $Y$.
*5)* $X$ *is a subspace of* $L_p(\mu)$ *for some Borel measure* $\mu$.

**Corollary 7** *If* $X^{**}$ *is a strong mid* $p$*-space then* $X$ *is a strong mid* $p$*-space.*

**Proof.** Is a direct consequence of Theorem 6 and [2, Proposition 2.8]. ■

**Theorem 8** *The following are equivalent :*
*1)* $X^*$ *is a strong mid* $p^*$*-space.*
*2)* $id_X \in \mathscr{D}_p^{mid}(X;X)$.
*3)* $\mathscr{L}(X;Y) = \mathscr{D}_p^{mid}(X;Y)$ *for every Banach space* $Y$.
*4)* $\mathscr{L}(Y;X) = \mathscr{D}_p^{mid}(Y;X)$ *for every Banach space* $Y$.
*5)* $id_{X^*} \in \Pi_{p^*}^{mid}(X^*;X^*)$.
*6)* $\mathscr{L}(X^*;Y) = \Pi_{p^*}^{mid}(X^*;Y)$ *for every Banach space* $Y$.
*7)* $\mathscr{L}(Y;X^*) = \Pi_{p^*}^{mid}(Y;X^*)$ *for every Banach space* $Y$.
*8)* $X^*$ *is a subspace of* $L_{p^*}(\mu)$ *for some Borel measure* $\mu$.

**Proof.** By Theorem 3 and Theorem 6, we have

$$\begin{array}{ccccccc} & & & & 7 & & \\ & & & & \Updownarrow & & \\ 8 & \Leftrightarrow & 1 & \Leftrightarrow & 5 & \Leftrightarrow & 6 \\ & & & & \Updownarrow & & \\ & & 3 & \Leftrightarrow & 2 & \Leftrightarrow & 4 \end{array}$$

■

**Theorem 9** *[2, Theorem 2.6][4, Theorem 3.7] The following are equivalent :*
*1)* $X$ *has the* $p$*-Dunford-Pettis property.*
*2)* $\Pi_p^{mid}(X;Y) = \Pi_p(X;Y)$ *for every Banach space* $Y$.
*3)* $\Pi_p^{mid}(X;\ell_p) = \Pi_p(X;\ell_p) = \mathscr{L}(X;\ell_p)$.
*4)* $\mathscr{D}_{p^*}(\ell_{p^*};X^*) = \mathscr{L}(\ell_{p^*};X^*)$.

**Theorem 10** *The following are equivalent :*
*1)* $X^*$ *has the p-Dunford-Pettis property.*
*2)* $\mathscr{D}_{p^*}^{mid}(Y;X) = \mathscr{D}_{p^*}(Y;X)$ *for every Banach space* $Y$.
*3)* $\Pi_p^{mid}(X^*;Y^*) = \Pi_p(X^*;Y^*)$ *for every Banach space* $Y$.
*4)* $\Pi_p^{mid}(X^*;\ell_p) = \Pi_p(X^*;\ell_p) = \mathscr{L}(X^*;\ell_p)$.
*5)* $\mathscr{D}_{p^*}^{mid}(\ell_{p^*};X) = \mathscr{D}_{p^*}(\ell_{p^*};X) = \mathscr{L}(\ell_{p^*};X)$.
*6)* $\mathscr{D}_{p^*}(\ell_{p^*};X) = \mathscr{L}(\ell_{p^*};X)$.

**Proof.** follows from Theorem 3 and Theorem 9. ■

**Corollary 11** $X^{**}$ *has the p-Dunford-Pettis property if and only if* $X$ *has the p-Dunford-Pettis property.*

**Corollary 12** *[4, Corollary 3.9] If* $X^*$ *has the p-Dunford-Pettis property, so does* $X$.

**Theorem 13** $X^*$ *has the p-Dunford-Pettis property if and only if* $X$ *has the p-Dunford-Pettis property.*

## 3. REFERENCES

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