EXISTENCE AND ULAM STABILITY OF K-GENERALIZED ψ-HILFER FRACTIONAL PROBLEM

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ABSTRACT

In this paper, we prove existence, uniqueness stability results for a class of initial value problem for fractional differential equations involving generalized ψ-Hilfer fractional derivative. The result is based on the Banach contraction mapping principle. In addition, two examples are given to illustrate our results.

1. INTRODUCTION

The differential and fractional integral calculus generalizes the notions of integration and derivation of integer order, its history dates back to 1695, see [1, 2, 3, 4, 8, 23] and the references therein. In these last few decades, the application of fractional differential calculus has been made in various fields of research and engineering during the last decades ; in control theory, bio-chemistry, economics, etc. There are various types of fractional derivatives such as, Riemann-Liouville fractional derivative, Caputo fractional derivative, Hadamard fractional derivative, Hilfer fractional derivative, Katugampola fractional derivative, Caputo-Fabrizio fractional derivative, Atangana-Baleanu-Caputo fractional derivative, ψ-fractional derivatives and many others, see [15, 16, 17, 18, 19, 9] and the references therein. Diaz *et al.* [6] have presented k-gamma and k-beta functions and demonstrated a number of their properties. Important properties can be found in the article [10], and very recently (see [5, 11, 12]), many researchers managed to generalize various fractional integrals and derivatives. Sousa and Capelas de Oliveira, in [21], introduce another so-called ψ-Hilfer fractional derivative with respect to a given function, and present some important properties concerning this type of fractional operator. They proved numerous results in [21, 20, 22].

Motivated by the works of the papers mentioned above, in this paper, by using the functions k-gamma, k-beta and k-Mittag-Leffler, we generalize the ψ-Hilfer fractional derivative and set some of its properties. Then, we propose a generalized Gronwall inequality, which will be used in an application. Finally, we consider the initial value problem with k-generalized ψ-Hilfer type fractional differential equation :

$$\left({}_k^H\mathscr{D}_{a+}^{\vartheta,r;\psi}x\right)(t) = f\left(t,x(t)\right), \ \ t\in(a,b], \tag{1}$$

$$\left(\mathscr{J}_{a+}^{k(1-\xi),k;\psi}x\right)(a^+) = x_0, \tag{2}$$

where ${}_k^H\mathscr{D}_{a+}^{\vartheta,r;\psi}, \mathscr{J}_{a+}^{k(1-\xi),k;\psi}$ are the k-generalize ψ-Hilfer fractional derivative of order $\vartheta\in(0,1)$ and type $r\in[0,1]$ defined in Section 2, and k-generalize ψ-fractional integral of order $k(1-\xi)$ defined in [13] respectively, where $\xi=\frac{1}{k}(r(k-\vartheta)+\vartheta),\ \theta<k,\ x_0\in\mathbb{R}$, $k>0$ and $f\in C([a,b]\times\mathbb{R},\mathbb{R})$.

The following are the primary novelties of the current paper :

1. Given that the ψ-Hilfer fractional derivative unifies a larger number of fractional derivatives in a single fractional operator, defining the k-generalized ψ-Hilfer fractional derivative allows us to encompass more fractional operators, opening the door to new applications.
2. The results of this study are generalizations of several results obtained in [21, 12, 1, 2, 3].

The present paper is organized as follows. In Section 2, some notations are introduced and we recall some preliminaries about ψ-Hilfer fractional derivative, the functions k-gamma, k-beta and k-Mittag-Leffler and some auxiliary results, then we define the k-generalize ψ-Hilfer type fractional derivative and give some necessary theorems and lemmas. In Section 3, a result for the problem (1)-(2) are presented which is based on Banach contraction principle. In Section 4, we study the Ulam-Hyers-Rassias (U-H-R) stability for our problem (1)-(2). Finally, in the last section, we give two examples to illustrate the applicability of our results.

2. PRELIMINARIES

First, we present the weighted spaces, notations, definitions, and preliminary facts which are used throughout this paper. Let $0 < a < b < \infty$, $J = [a,b]$, $\vartheta \in (0,1)$, $r \in [0,1]$, $k > 0$ and $\xi = \frac{1}{k}(r(k-\vartheta)+\vartheta)$. By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|x\|_\infty = \sup\{|x(t)| : t \in J\}.$$

$AC^n(J,\mathbb{R})$, $C^n(J,\mathbb{R})$ be the spaces of continuous functions, n-times absolutely continuous and n-times continuously differentiable functions on J, respectively.
Consider the weighted Banach space

$$C_{\xi;\psi}(J) = \left\{x : (a,b] \to \mathbb{R} : t \to (\psi(t)-\psi(a))^{1-\xi}\, x(t) \in C(J,\mathbb{R})\right\},$$

with the norm

$$\|x\|_{C_{\xi;\psi}} = \sup_{t\in J}\left|(\psi(t)-\psi(a))^{1-\xi}\, x(t)\right|,$$

and

$$C^n_{\xi;\psi}(J) = \left\{x \in C^{n-1}(J) : x^{(n)} \in C_{\xi;\psi}(J)\right\}, n \in \mathbb{N},$$
$$C^0_{\xi;\psi}(J) = C_{\xi;\psi}(J),$$

with the norm

$$\|x\|_{C^n_{\xi;\psi}} = \sum_{i=0}^{n-1} \|x^{(i)}\|_\infty + \|x^{(n)}\|_{C_{\xi;\psi}}.$$

Consider the space $X^p_\psi(a,b)$, $(c \in \mathbb{R},\ 1 \le p \le \infty)$ of those real-valued Lebesgue measurable functions g on $[a,b]$ for which $\|g\|_{X^p_\psi} < \infty$, where the norm is defined by

$$\|g\|_{X^p_\psi} = \left(\int_a^b \psi'(t)|g(t)|^p dt\right)^{\frac{1}{p}},$$

where ψ is an increasing and positive function on $[a,b]$ such that ψ' is continuous on $[a,b]$ with $\psi(0) = 0$. In particular, when $\psi(x) = x$, the space $X^p_\psi(a,b)$ coincides with the $L_p(a,b)$ space. Recently, in [6], Diaz and Petruel have defined new functions called k-gamma and k-beta functions given by

$$\Gamma_k(\alpha) = \int_0^\infty t^{\alpha-1} e^{-\frac{t^k}{k}} dt, \alpha > 0.$$

• For $k \to 1 : \Gamma_k(\alpha) \to \Gamma(\alpha)$.
• We have :

$$\Gamma_k(\alpha) = k^{\frac{\alpha}{k}-1}\Gamma\left(\frac{\alpha}{k}\right),\ \Gamma_k(\alpha+k) = \alpha\Gamma_k(\alpha),\ \Gamma_k(k) = \Gamma(1) = 1.$$

Furthermore k-beta function is defined as follows :

$$B_k(\alpha,\beta) = \frac{1}{k}\int_0^1 t^{\frac{\alpha}{k}-1}(1-t)^{\frac{\beta}{k}-1}dt$$

so that

$$B_k(\alpha,\beta) = \frac{1}{k}B\left(\frac{\alpha}{k},\frac{\beta}{k}\right),\ B_k(\alpha,\beta) = \frac{\Gamma_k(\alpha)\Gamma_k(\beta)}{\Gamma_k(\alpha+\beta)}.$$

The Mittag-Leffler function can also be refined into the k-Mittag-Leffler function defined as follows

$$E_k^{\alpha,\beta}(x) = \sum_{i=0}^{\infty}\frac{x^i}{\Gamma_k(\alpha i+\beta)}, \alpha,\beta > 0.$$

Now, we give all the definitions to the different fractional operators used throughout this paper.

Definition 1 *([13]) (k-Generalized ψ-fractional Integral) Let $g \in X_\psi^p(a,b)$ and $[a,b]$ be a finite or infinite interval on the real axis $\mathbb{R} = (-\infty,\infty)$, $\psi(t) > 0$ be an increasing function on $(a,b]$ and $\psi'(t) > 0$ be continuous on (a,b) and $\vartheta > 0$. The generalized k-fractional integral operator of a function f (left-sided) of order ϑ is defined by*

$$\mathscr{I}_{a+}^{\vartheta,k;\psi}g(t) = \frac{1}{k\Gamma_k(\vartheta)}\int_a^t \frac{\psi'(s)g(s)ds}{(\psi(t)-\psi(s))^{1-\frac{\vartheta}{k}}},$$

with $k > 0$.

Remark 1 *Nápoles Valdés ([11]) gave a more generalized fractional integral operators defined by*

$$\mathscr{I}_{G,a+}^{\vartheta,k;\psi}g(t) = \frac{1}{k\Gamma_k(\vartheta)}\int_a^t \frac{\psi'(s)g(s)ds}{G(\psi(t)-\psi(s),\frac{\vartheta}{k})},$$

where $G(z,\vartheta) \in AC[a,b]$.

Theorem 1 *([11]) Let $g : [a,b] \to \mathbb{R}$ be an integrable function, and take $\vartheta > 0$ and $k > 0$. Then $\mathscr{I}_{G,a+}^{\vartheta,k;\psi}g$ exists for all $t \in [a,b]$.*

Theorem 2 *([11]) Let $g \in X_\psi^p(a,b)$ and take $\vartheta > 0$ and $k > 0$. Then $\mathscr{I}_{G,a+}^{\vartheta,k;\psi}g \in C([a,b],\mathbb{R})$.*

Lemma 3 *[14] Let $\vartheta > 0$, $r > 0$ and $k > 0$. Then, we have the following semigroup property given by*

$$\mathscr{I}_{a+}^{\vartheta,k;\psi}\mathscr{I}_{a+}^{r,k;\psi}f(t) = \mathscr{I}_{a+}^{\vartheta+r,k;\psi}f(t) = \mathscr{I}_{a+}^{r,k;\psi}\mathscr{I}_{a+}^{\vartheta,k;\psi}f(t).$$

Lemma 4 *[14] Let $\vartheta, r > 0$ and $k > 0$. Then, we have*

$$\mathscr{I}_{a+}^{\vartheta,k;\psi}[\psi(t)-\psi(a)]^{\frac{r}{k}-1} = \frac{\Gamma_k(r)}{\Gamma_k(\vartheta+r)}(\psi(t)-\psi(a))^{\frac{\vartheta+r}{k}-1}.$$

Theorem 5 *[14] Let* $0 < a < b < \infty, \vartheta > 0, 0 \leq \xi < 1$, $k > 0$ *and* $x \in C_{\xi;\psi}(J)$*. If* $\frac{\vartheta}{k} > 1 - \xi$*, then*

$$\left(\mathscr{J}_{a^+}^{\vartheta,k;\psi} x\right)(a) = \lim_{t \to a^+} \left(\mathscr{J}_{a^+}^{\vartheta,k;\psi} x\right)(t) = 0.$$

We are now able to define the k-generalized ψ-Hilfer derivative as follows.

Definition 2 *[14] (2021) (k-Generalized* ψ*-Hilfer Derivative) Let* $n - 1 < \frac{\vartheta}{k} \leq n$ *with* $n \in \mathbb{N}$*,* $J = [a,b]$ *an interval such that* $-\infty \leq a < b \leq \infty$ *and* $g, \psi \in C^n([a,b], \mathbb{R})$ *two functions such that* ψ *is increasing and* $\psi'(t) \neq 0$*, for all* $t \in J$*. The k-generalized* ψ*-Hilfer fractional derivative (left-sided)* ${}_k^H\mathscr{D}_{a+}^{\vartheta,r;\psi}(\cdot)$ *of a function g of order* ϑ *and type* $0 \leq r \leq 1$*, with* $k > 0$ *is defined by*

$$\begin{aligned} {}_k^H\mathscr{D}_{a+}^{\vartheta,r;\psi} g(t) &= \left(\mathscr{J}_{a+}^{r(kn-\vartheta),k;\psi} \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n \left(k^n \mathscr{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi} g\right)\right)(t) \\ &= \left(\mathscr{J}_{a+}^{r(kn-\vartheta),k;\psi} \delta_\psi^n \left(k^n \mathscr{J}_{a+}^{(1-r)(kn-\vartheta),k;\psi} g\right)\right)(t), \end{aligned}$$

where $\delta_\psi^n = \left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n$.

Lemma 6 *[14] Let* $t > a$, $\vartheta > 0, 0 \leq r \leq 1, k > 0$*. Then for* $0 < \xi < 1; \xi = \frac{1}{k}(r(k - \vartheta) + \vartheta)$*, we have*

$$\left[{}_k^H\mathscr{D}_{a+}^{\vartheta,r;\psi} (\psi(s) - \psi(a))^{\xi-1}\right](t) = 0.$$

Theorem 7 *[14] If* $f \in C_{\xi;\psi}^n[a,b], n - 1 < \vartheta < n$*,* $0 \leq r \leq 1$*, where* $n \in \mathbb{N}$ *and* $k > 0$*, then*

$$\left(\mathscr{J}_{a+}^{\vartheta,k;\psi}\, {}_k^H\mathscr{D}_{a+}^{\vartheta,r;\psi} f\right)(t) = f(t) - \sum_{i=1}^{n} \frac{(\psi(t) - \psi(a))^{\xi-i}}{k^{i-n}\Gamma_k(k(\xi - i + 1))} \left\{\delta_\psi^{n-i}\left(\mathscr{J}_{a+}^{k(n-\xi),k;\psi} f(a)\right)\right\},$$

where

$$\xi = \frac{1}{k}(r(kn - \vartheta) + \vartheta).$$

- *For* $n = 1$ *:*

$$\left(\mathscr{J}_{a+}^{\vartheta,k;\psi}\, {}_k^H\mathscr{D}_{a+}^{\vartheta,r;\psi} f\right)(t) = f(t) - \frac{(\psi(t) - \psi(a))^{\xi-1}}{\Gamma_k(r(k - \vartheta) + \vartheta)} \mathscr{J}_{a+}^{(1-r)(k-\vartheta),k;\psi} f(a).$$

Lemma 8 *[14] Let* $\vartheta(0,), 0 \leq r \leq 1$*, and* $x \in C_{\xi;\psi}^1(J)$*, where* $k > 0$*, then for* $t \in (a,b]$*, we have*

$$\left({}_k^H\mathscr{D}_{a+}^{\vartheta,r;\psi}\ \mathscr{J}_{a+}^{\vartheta,k;\psi} x\right)(t) = x(t).$$

Now, we consider the Ulam stability for problem $(1)-(2)$ that will be used in Section 4. Let $x \in C_{\xi;\psi}^1(J)$, $\varepsilon > 0$ and $v : (a,b] \longrightarrow [0,\infty)$ be a continuous function. We consider the following inequality :

$$\left|\left({}_k^H\mathscr{D}_{a+}^{\vartheta,r;\psi} x\right)(t) - f(t, x(t))\right| \leq \varepsilon v(t),\ t \in (a,b]. \tag{3}$$

Definition 3 *Problem (1)−(2) is Ulam-Hyers-Rassias (U-H-R) stable with respect to v if there exists a real number $a_{f,v} > 0$ such that for each $\varepsilon > 0$ and for each solution $x \in C^1_{\xi;\psi}(J)$ of inequality (3) there exists a solution $y \in C^1_{\xi;\psi}(J)$ of (1)−(2) with*

$$|x(t) - y(t)| \leq \varepsilon a_{f,v} v(t), \qquad t \in J.$$

Remark 2 *A function $x \in C^1_{\xi;\psi}(J)$ is a solution of inequality (3) if and only if there exist $\sigma \in C_{\xi;\psi}(J)$ such that*

1. $|\sigma(t)| \leq \varepsilon v(t)$, $t \in (a,b]$,
2. $\left({}^H_k\mathscr{D}^{\vartheta,r;\psi}_{a+} x\right)(t) = f(t, x(t)) + \sigma(t)$, $t \in (a,b]$.

We give a generalized Gronwall inequality which will be used in Section 4.

Lemma 9 (The Gronwall inequality) *[14] (2021) Let x, y be two integrable functions and g continuous, with domain $[a,b]$. Let $\psi \in C^1[a,b]$ an increasing function such that $\psi'(t) \neq 0$, $t \in [a,b]$ and $\vartheta > 0$ with $k > 0$. Assume that*

1. *x and y are nonnegative ;*
2. *w is nonnegative and nondecreasing.*

• *If*

$$x(t) \leq y(t) + \frac{w(t)}{k} \int_a^t \psi'(s) [\psi(t) - \psi(s)]^{\frac{\vartheta}{k}-1} x(s)\, ds,$$

then

$$x(t) \leq y(t) + \int_a^t \sum_{i=1}^{\infty} \frac{[w(t)\Gamma_k(\vartheta)]^i}{k\Gamma_k(\vartheta i)} \psi'(s) [\psi(t) - \psi(s)]^{\frac{i\vartheta}{k}-1} y(s)\, ds, \tag{4}$$

for all $t \in [a,b]$.
• *If y is a nondecreasing function on $[a,b]$, then,*

$$x(t) \leq y(t) \mathbb{E}^{\vartheta,k}_k \left(w(t) \Gamma_k(\vartheta) (\psi(t) - \psi(a))^{\frac{\vartheta}{k}} \right).$$

3. EXISTENCE OF SOLUTIONS

We consider the following fractional differential equation

$$\left({}^H_k\mathscr{D}^{\vartheta,r;\psi}_{a+} x\right)(t) = w(t), \;\; t \in (a,b], \tag{5}$$

where $0 < \vartheta < 1, 0 \leq r \leq 1$, with the condition

$$\left(\mathscr{J}^{k(1-\xi),k;\psi}_{a+} x\right)(a^+) = x_0, \tag{6}$$

where $\xi = \dfrac{r(k-\vartheta)+\vartheta}{k}$, $x_0 \in \mathbb{R}$, $k > 0$, and where $w \in C(J,\mathbb{R})$ satisfies the functional equation :

$$w(t) = f(t, x(t)).$$

The following theorem shows that the problem (5)-(6) have a unique solution.

Theorem 10 *If $w(\cdot) \in C^1_{\xi;\psi}(J)$, then x satisfies (5)-(6) if and only if it satisfies*

$$x(t) = \frac{(\psi(t) - \psi(a))^{\xi-1}}{\Gamma_k(k\xi)} x_0 + \left(\mathscr{J}^{\vartheta,k;\psi}_{a+} w\right)(t). \tag{7}$$

Proof. Assume $x \in C^1_{\xi;\psi}(J)$ satisfies the equations (5) and (6), and applying the fractional integral operator $\mathscr{I}^{\vartheta,k;\psi}_{a+}(\cdot)$ on both sides of the fractional equation (5), so

$$\left(\mathscr{I}^{\vartheta,k;\psi}_{a+}\,{}^{H}_{k}\mathscr{D}^{\vartheta,r;\psi}_{a+}x\right)(t) = \left(\mathscr{I}^{\vartheta,k;\psi}_{a+}w\right)(t),$$

and using Theorem 7 and equation (6), we get

$$\begin{aligned} x(t) &= \frac{(\psi(t)-\psi(a))^{\xi-1}}{\Gamma_k(k\xi)}\mathscr{I}^{k(1-\xi),k;\psi}_{a+}x(a) + \left(\mathscr{I}^{\vartheta,k;\psi}_{a+}w\right)(t) \\ &= \frac{(\psi(t)-\psi(a))^{\xi-1}}{\Gamma_k(k\xi)}x_0 + \left(\mathscr{I}^{\vartheta,k;\psi}_{a+}w\right)(t). \end{aligned}$$

Let us now prove that if x satisfies equation (7), then it satisfies equations (5) and (6). Applying the fractional derivative operator ${}^{H}_{k}\mathscr{D}^{\vartheta,r;\psi}_{a+}(\cdot)$ on both sides of the fractional equation (7), then we get

$$\left({}^{H}_{k}\mathscr{D}^{\vartheta,r;\psi}_{a+}x\right)(t) = {}^{H}_{k}\mathscr{D}^{\vartheta,r;\psi}_{a+}\left(\frac{(\psi(t)-\psi(a))^{\xi-1}}{\Gamma_k(k\xi)}x_0\right) + \left({}^{H}_{k}\mathscr{D}^{\vartheta,r;\psi}_{a+}\,\mathscr{I}^{\vartheta,k;\psi}_{a+}w\right)(t).$$

Using the Lemma 6 and Lemma 8, we obtain equation (5). Now we apply the operator $\mathscr{I}^{k(1-\xi),k;\psi}_{a+}(\cdot)$ on equation (7), to have

$$\left(\mathscr{I}^{k(1-\xi),k;\psi}_{a+}x\right)(t) = \frac{x_0}{\Gamma_k(k\xi)}\mathscr{I}^{k(1-\xi),k;\psi}_{a+}(\psi(t)-\psi(a))^{\xi-1} + \left(\mathscr{I}^{k(1-\xi),k;\psi}_{a+}\,\mathscr{I}^{\vartheta,k;\psi}_{a+}w\right)(t).$$

Now, using Lemma 3 and 4, we get

$$\begin{aligned} \left(\mathscr{I}^{k(1-\xi),k;\psi}_{a+}x\right)(t) &= \frac{x_0}{\Gamma_k(k\xi)}\mathscr{I}^{k(1-\xi),k;\psi}_{a+}(\psi(t)-\psi(a))^{\xi-1} + \left(\mathscr{I}^{k(1-\xi),k;\psi}_{a+}\,\mathscr{I}^{\vartheta,k;\psi}_{a+}w\right)(t) \\ &= x_0 + \left(\mathscr{I}^{k(1-\xi)+\vartheta,k;\psi}_{a+}w\right)(t). \end{aligned}$$

Using Theorem 5 with $t \to a$, we obtain equation (6). This complete the proof. ∎ As a consequence of Theorem 10, we have the following result :

Lemma 11 *Let* $\xi = \dfrac{r(k-\vartheta)+\vartheta}{k}$ *where* $0<\vartheta<1$, $0\le r\le 1$ *and* $k>0$, *let* $f: J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *be a continuous function such that* $f(\cdot,x(\cdot))\in C^1_{\xi;\psi}(J)$, *for any* $x,y\in C_{\xi;\psi}(J)$. *If* $x\in C^1_{\xi;\psi}(J)$, *then* x *satisfies the problem* $(1)-(2)$ *if and only if* x *is the fixed point of the operator* $\mathscr{T}:C_{\xi;\psi}(J)\to C_{\xi;\psi}(J)$ *defined by*

$$(\mathscr{T}x)(t) = \frac{(\psi(t)-\psi(a))^{\xi-1}}{\Gamma_k(k\xi)}x_0 + \frac{1}{k\Gamma_k(\vartheta)}\int_a^t \frac{\psi'(s)\varphi(s)ds}{(\psi(t)-\psi(s))^{1-\frac{\vartheta}{k}}}. \tag{8}$$

where φ *be a function satisfying the functional equation*

$$\varphi(t) = f(t,x(t)).$$

The following hypotheses will be used in the sequel :
(*Cd.1*) The function $f: J\times\mathbb{R}\to\mathbb{R}$ is continuous and
$$f(\cdot,x(\cdot))\in C^1_{\xi;\psi}(J), \text{ for any } x\in C_{\xi;\psi}(J).$$

($Cd.2$) There exists constant $\eta > 0$ such that

$$|f(t,x) - f(t,\bar{x})| \leq \eta |x - \bar{x}|$$

for any $x, \bar{x} \in \mathbb{R}$ and $t \in J$.

We are now in a position to state and prove our existence result for the problem (1)-(2) based on Banach's fixed point theorem.

Theorem 12 *Assume* (Cd.1) *and* (Cd.2) *hold. If*

$$\mathscr{L} = \frac{\eta \Gamma_k(k\xi)\left(\psi(b) - \psi(a)\right)^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k\xi)} < 1, \tag{9}$$

then the problem (1)-(2) has a unique solution in $C_{\xi;\psi}(J)$.

Proof. We show that the operator $\mathscr{T}$ defined in (8) has a unique fixed point in $C_{\xi;\psi}(J)$. Let $x, y \in C_{\xi;\psi}(J)$ and $t \in (a,b]$. Then, for $t \in J$ we have

$$|\mathscr{T}x(t) - \mathscr{T}y(t)| \leq \frac{1}{k\Gamma_k(\vartheta)} \int_a^t \frac{\psi'(s)|\varphi_1(s) - \varphi_2(s)|dt}{(\psi(t) - \psi(s))^{1-\frac{\vartheta}{k}}},$$

where φ_1 and φ_1 be functions satisfying the functional equations

$$\varphi_1(t) = f(t, x(t)),$$
$$\varphi_2(t) = f(t, y(t)).$$

By ($Cd.2$), we have

$$\begin{aligned} |\varphi_1(t) - \varphi_2(t)| &= |f(t,x(t)) - f(t,y(t))| \\ &\leq \eta |x(t) - y(t)|. \end{aligned}$$

Therefore, for each $t \in (a,b]$

$$\begin{aligned} |\mathscr{T}x(t) - \mathscr{T}y(t)| &\leq \frac{\eta}{k\Gamma_k(\vartheta)} \int_a^t \frac{\psi'(s)|x(s) - y(s)|dt}{(\psi(t) - \psi(s))^{1-\frac{\vartheta}{k}}} \\ &\leq \eta \|x - y\|_{C_{\xi;\psi}} \mathscr{I}_{a+}^{\vartheta,k;\psi} (\psi(t) - \psi(a))^{\xi - 1}. \end{aligned}$$

By Lemma 4, we have

$$|\mathscr{T}x(t) - \mathscr{T}y(t)| \leq \left[\frac{\eta \Gamma_k(k\xi)}{\Gamma_k(\vartheta + k\xi)} \left(\psi(t) - \psi(a)\right)^{\frac{\vartheta + k\xi}{k} - 1}\right] \|x - y\|_{C_{\xi;\psi}},$$

hence

$$\begin{aligned} \left|(\psi(t) - \psi(a))^{1-\xi} \left(\mathscr{T}x(t) - \mathscr{T}y(t)\right)\right| &\leq \left[\frac{\eta \Gamma_k(k\xi)\left(\psi(t) - \psi(a)\right)^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k\xi)}\right] \|x - y\|_{C_{\xi;\psi}} \\ &\leq \left[\frac{\eta \Gamma_k(k\xi)\left(\psi(b) - \psi(a)\right)^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k\xi)}\right] \|x - y\|_{C_{\xi;\psi}}, \end{aligned}$$

which implies that

$$\|\mathscr{T}x - \mathscr{T}y\|_{C_{\xi;\psi}} \leq \left[\frac{\eta \Gamma_k(k\xi)\left(\psi(b) - \psi(a)\right)^{\frac{\vartheta}{k}}}{\Gamma_k(\vartheta + k\xi)}\right] \|x - y\|_{C_{\xi;\psi}}.$$

By (9), the operator $\mathscr{T}$ is a contraction. Hence, by Banach's contraction principle, $\mathscr{T}$ has a unique fixed point $x \in C_{\xi;\psi}(J)$, which is a solution to our problem (1)-(2). ■

4. ULAM-HYERS-RASSIAS STABILITY

Theorem 13 *Assume that in addition to (*Cd.1*),(*Cd.2*) and (9), the following hypothesis holds.*

(Cd.3) *There exist a nondecreasing function $v \in C^1_{\xi;\psi}(J)$ and $\kappa_v > 0$ such that for each $t \in J$, we have*

$$\left(\mathscr{J}_{a+}^{\vartheta,k;\psi} v\right)(t) \leq \kappa_v v(t),$$

Then the problem (1)-(2) is U-H-R stable with respect to v.

Proof. Let $x \in C^1_{\xi;\psi}(J)$ be a solution if inequality (3), and let us assume that y is the unique solution of the problem

$$\begin{cases} \left({}_k^H\mathscr{D}_{a+}^{\vartheta,r;\psi} y\right)(t) = f(t,y(t))\,;\ t \in (a,b], \\ \left(\mathscr{J}_{a+}^{k(1-\xi),k;\psi} y\right)(a^+) = \left(\mathscr{J}_{a+}^{k(1-\xi),k;\psi} x\right)(a^+). \end{cases}$$

By Lemma 11, we obtain for each $t \in (a,b]$

$$y(t) = \frac{(\psi(t)-\psi(a))^{\xi-1}}{\Gamma_k(k\xi)} \mathscr{J}_{a+}^{k(1-\xi),k;\psi} y(a) + \left(\mathscr{J}_{a+}^{\vartheta,k;\psi} w\right)(t),$$

where $w \in C^1_{\xi;\psi}(J)$, be a function satisfying the functional equation

$$w(t) = f(t,y(t)).$$

Since x is a solution of the inequality (3), by Remark 2, we have

$$\left({}_k^H\mathscr{D}_{a+}^{\vartheta,r;\psi} x\right)(t) = f(t,x(t)) + \sigma(t), t \in (a,b]. \tag{10}$$

Clearly, the solution of (10) is given by

$$x(t) = \frac{(\psi(t)-\psi(a))^{\xi-1}}{\Gamma_k(k\xi)} \mathscr{J}_{a+}^{k(1-\xi),k;\psi} x(a) + \left(\mathscr{J}_{a+}^{\vartheta,k;\psi} (\tilde{w}+\sigma)\right)(t),$$

where $\tilde{w} \in C^1_{\xi;\psi}(J)$ be a function satisfying the functional equation

$$\tilde{w}(t) = f(t,x(t)).$$

Hence, for each $t \in (a,b]$, we have

$$\begin{aligned} |x(t)-y(t)| &\leq \left(\mathscr{J}_{a+}^{\vartheta,k;\psi} |\tilde{w}(s)-w(s)|\right)(t) + \left(\mathscr{J}_{a+}^{\vartheta,k;\psi} \sigma\right)(t) \\ &\leq \varepsilon\kappa_v v(t) + \frac{\eta}{k\Gamma_k(\vartheta)} \int_a^t \frac{\psi'(s)|x(s)-y(s)|dt}{(\psi(t)-\psi(s))^{1-\frac{\vartheta}{k}}}. \end{aligned}$$

By applying Lemma 9, we obtain

$$\begin{aligned} |x(t)-y(t)| &\leq \varepsilon\kappa_v v(t) + \int_a^t \sum_{i=1}^{\infty} \frac{(\eta)^i}{k\Gamma_k(\vartheta i)} \psi'(s)\left[\psi(t)-\psi(s)\right]^{\frac{i\vartheta}{k}-1} \varepsilon\kappa_v v(s)\,ds, \\ &\leq \varepsilon\kappa_v v(t) \mathbb{E}_k^{\vartheta,k}\left[\eta\left(\psi(t)-\psi(a)\right)^{\frac{\vartheta}{k}}\right] \\ &\leq \varepsilon\kappa_v v(t) \mathbb{E}_k^{\vartheta,k}\left[\eta\left(\psi(b)-\psi(a)\right)^{\frac{\vartheta}{k}}\right]. \end{aligned}$$

Then for each $t \in (a,b]$, we have

$$|x(t)-y(t)| \leq a_{f,v}\varepsilon v(t),$$

where

$$a_{f,v} = \kappa_v \mathbb{E}_k^{\vartheta,k}\left[\eta\left(\psi(b)-\psi(a)\right)^{\frac{\vartheta}{k}}\right].$$

Hence, the problem (1)-(2) is U-H-R stable with respect to v. ■

5. EXAMPLES

With the following examples, we look at particular cases of the problem (1)-(2).

Example 1 *Taking $r \to 0$, $\vartheta = \frac{1}{2}$, $k = 1$, $\psi(t) = t$, $a = 1$, $b = 2$ and $x_0 = 1$, we obtain a particular case of problem* (1)-(2) *with Riemann-Liouville fractional derivative, given by*

$$\left({}^H_1\mathscr{D}^{\frac{1}{2},0;\psi}_{1+} x\right)(t) = \left({}^{RL}\mathbb{D}^{\frac{1}{2}}_{1^+} x\right)(t) = f(t, x(t)), \ \ t \in (1,2], \tag{11}$$

$$\left(\mathscr{J}^{\frac{1}{2},1;\psi}_{1+} x\right)(1^+) = 1, \tag{12}$$

where $J = [1,2]$, $\xi = \frac{1}{k}(r(k-\vartheta)+\vartheta) = \frac{1}{2}$ and

$$f(t,x) = \frac{\sqrt{t-1}|sin(t)|(1+x)}{120e^{-t+3}}, \ t \in J, \ x \in \mathbb{R}.$$

We have

$$C_{\xi;\psi}(J) = C_{\frac{1}{2};\psi}(J) = \left\{u : (1,2] \to \mathbb{R} : (\sqrt{t-1})u \in C(J,\mathbb{R})\right\},$$

and

$$C^1_{\xi;\psi}(J) = C^1_{\frac{1}{2};\psi}(J) = \left\{u \in C_{\frac{1}{2};\psi}(J) : u' \in C_{\frac{1}{2};\psi}(J)\right\},$$

*Since the continuous function $f \in C^1_{\frac{1}{2};\psi}(J)$, then the condition (*Cd.1*) is satisfied.*
For each $x, y \in \mathbb{R}$ and $t \in J$, we have

$$|f(t,x) - f(t,y)| \leq \frac{\sqrt{t-1}|sin(t)|}{120e^{-t+3}}|x-y|, \ t \in J,$$

*and so the condition (*Cd.2*) is satisfied with $\eta = \dfrac{1}{120e}$. Also, the condition (9) of Theorem 12 is satisfied. Indeed, we have*

$$\mathscr{L} = \frac{\sqrt{\pi}}{120e} \approx 0,00543374443477744 < 1.$$

Then the problem (11)−(12) has a unique solution in $C^1_{\frac{1}{2};\psi}([1,2])$.
Now, if we take $v(t) = t-1$ and $\kappa_v = \dfrac{\sqrt{2}\Gamma(2)}{\Gamma(\frac{5}{2})}$, then for each $t \in J$, we get

$$\begin{aligned} \left(\mathscr{J}^{\frac{1}{2},1;\psi}_{1+} v\right)(t) &\leq \frac{\sqrt{2}\Gamma(2)}{\Gamma(\frac{5}{2})}(t-1) \\ &= \kappa_v v(t), \end{aligned}$$

*which shows that the hypothesis (*Cd.3*) is satisfied. Consequently, Theorem 13 implies that the problem (11)-(12) is U-H-R stable.*

Example 2 *Taking $r \to 0$, $\vartheta = \frac{1}{2}$, $k = 1$, $\psi(t) = \ln t$, $a = 1$, $b = e$ and $x_0 = \pi$, we get a particular case of problem* $(1)-(2)$ *using the Hadamard fractional derivative, given by*

$$\left({}^H_1\mathscr{D}^{\frac{1}{2},0;\psi}_{1+} x\right)(t) = \left({}^{HD}\mathbb{D}^{\frac{1}{2}}_{1^+} x\right)(t) = f(t, x(t)), \ \ t \in (1,e], \tag{13}$$

$$\left(\mathscr{J}_{1^+}^{\frac{1}{2},1;\psi}x\right)(1^+) = \pi, \tag{14}$$

where $J = [1,e]$*, and*

$$f(t,x,y) = \frac{e+x}{222e^t},\ t \in J,\ x \in \mathbb{R}.$$

We have

$$C_{\xi;\psi}(J) = C_{\frac{1}{2};\psi}(J) = \left\{u : (1,e] \to \mathbb{R} : (\sqrt{\ln t})u \in C(J,\mathbb{R})\right\},$$

and

$$C^1_{\xi;\psi}(J) = C^1_{\frac{1}{2};\psi}(J) = \left\{u \in C_{\frac{1}{2};\psi}(J) : u' \in C_{\frac{1}{2};\psi}(J)\right\},$$

Clearly, the function $f \in C^1_{\frac{1}{2};\psi}(J)$*. Hence condition (*Cd.1*) is satisfied.*
For each $x, y, \in \mathbb{R}$ *and* $t \in J$*, we have*

$$|f(t,x) - f(t,y)| \le \frac{1}{222e^t}|x-y|,\ t \in J,$$

*and so the condition (*Cd.2*) is satisfied with* $\eta = \dfrac{1}{222e}$.
Also, we have

$$\mathscr{L} = \frac{\sqrt{\pi}}{222e} \approx 0.00293715915393375\ < 1,$$

then, the condition (9) of Theorem 12 is satisfied. Then the problem (13)–(14) has a unique solution in $C_{\frac{1}{2};\psi}([1,e])$*. The problem is also U-H-R stable if we take* $v(t) = e^2$ *and* $\kappa_v = \dfrac{1}{\Gamma(\frac{3}{2})}$*.*
Indeed, for each $t \in J$*, we get*

$$\begin{aligned}\left(\mathscr{J}_{1^+}^{\frac{1}{2},1;\psi}v\right)(t) &\le \frac{e^2}{\Gamma(\frac{3}{2})}\\ &= \kappa_v v(t).\end{aligned}$$

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