A FUNCTIONAL EQUATION ARISING IN DYNAMIC PROGRAMMING VIA A GENERALIZED F- WEAK CONTRACTIONS OF HARDY-ROGERS

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ABSTRACT

In this work, we define the notion of generalized F-weak contraction and prove the corresponding fixed point theorem. Roughly speaking, we generalize and improve the notion of F-contraction of Hardy-Rogers-type in the setting of complete b-metric spaces. As an application we establish sufficient criteria for the existence and uniqueness of bounded solutions of functional equations in dynamic programming.

Keywords :b-Metric space, contraction of Hardy–Rogers type, dynamic programming, F-contraction, fixed point, functional equations.

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1. INTRODUCTION

The well known Banach's contraction mapping principle [2] is the most significant fundamental fixed point result. Since this principle has a lot of applications in different branches of mathematics, several authors have extended, generalized and improved it in many directions by considering different forms of mappings or various types of spaces. In the paper [15], an interesting generalization of Banach contraction principle is given by introducing the concept of F-contraction. After that, the notion of F-weak contraction of Hardy-Rogers is introduced in [11] as a generalization of F-contraction in complete metric spaces. One of the most prevalent generalization of the metric spaces was given in the article [4] through the notion of b-metric spaces. In our paper, we utilize these two last notions to introduce new types of F-weak contractions of Hardy-Rogers in the setting of b-metric spaces and to prove some fixed point results. Roughly speaking, we extend and improve (respectively, improve) some results in [11]. More precisely, the results obtained in our paper extended the aforementioned results in b-metric spaces and contain less conditions imposed on the function F. Moreover, the consequences of our main results are improved and generalized versions of some results appearing in literature.

2. F-WEAK CONTRACTIONS OF HARDY-ROGERS TYPE

In the first, we recall some known definitions and results which will be used in the sequel.

2.1. b-metric spaces

In 1989, Bakhtin [4] introduced the concept of b-metric spaces as follows

Definition 2.1 *Let X be a nonempty set and let $s \geq 1$ be a given real number. A mapping σ : $X \times X \to [0,\infty)$ is said to be a b-metric if, for all $x,y,z \in X$, the following conditions hold :*

(b_1) $\sigma(x,y)=0$ *if and only if* $x=y$;

(b_2) $\sigma(x,y)=\sigma(y,x)$;

(b_3) $\sigma(x,z)\leq s[\sigma(x,y)+\sigma(y,z)]$.

Example 2.2 *Let (X,d) be a metric space and let the mapping $\sigma_d : X\times X\to[0,\infty)$ be defined by*

$$\sigma_d(x,y)=(d(x,y))^p, \quad \text{for all } x,y\in X,$$

where $p>1$ is a fixed real number. Then (X,σ_d) is a b-metric space with $s=2^{p-1}$.

We present now the concepts of convergence, Cauchy sequence and completeness in b-metric spaces.

Definition 2.3 *Let (X,σ) be a b-metric space. Then a sequence $\{x_n\}$ in X is called :*

(a) *convergent if and only if there exists $x\in X$ such that $\lim\limits_{n\to\infty}\sigma(x_n,x)=0$ and in this case we write $\lim\limits_{n\to\infty}x_n=x$;*

(b) *Cauchy if and only if $\lim\limits_{n,m\to\infty}\sigma(x_n,x_m)=0$.*

Definition 2.4 *The b-metric space (X,σ) is said complete if every Cauchy sequence in X converges in X.*

Remark 2.5 *In a b-metric space, the following assertions hold :*

(i) *a convergent sequence has a unique limit;*

(ii) *each convergent sequence is Cauchy.*

2.2. F-weak contraction of Hardy-Rogers type

In 2012, Wardowski [15] introduced the notion of F-contraction as follows :

Definition 2.6 ***(See** [15]**)** Let (X,d) be a metric space. A mapping $T:X\to X$ is said to be an F-contraction if there exist $F\in\mathscr{F}$ and $\tau>0$ such that for all $x,y\in X$,*

$$d(Tx,Ty)>0\Rightarrow \tau+F(d(Tx,Ty))\leq F(d(x,y)), \tag{1}$$

where $\mathscr{F}$ is the family of all functions $F:(0,\infty)\to\mathbb{R}$ satisfying the following conditions :

(F_1) *F is strictly increasing, i.e., for all $\alpha,\beta\in(0,\infty)$, if $\alpha<\beta$, then $F(\alpha)<F(\beta)$.*

(F_2) *For each sequence $\{\alpha_n\}$ of positive numbers, the following holds :*

$$\lim_{n\to\infty}\alpha_n=0 \text{ if and only if } \lim_{n\to\infty}F(\alpha_n)=-\infty.$$

(F_3) *There exists $k\in(0,1)$ such that $\lim\limits_{\alpha\to0^+}\alpha^kF(\alpha)=0$.*

Lukács and Kajántó [11] defined a new class of functions (noted $\mathscr{F}_{s,\tau}$). Their definition is given below.

Definition 2.7 ***(See** [11, Definition 2.7]**)** Let $s\geq1$ and $\tau>0$. We say that $F\in\mathbb{F}^*$ belongs to $\mathscr{F}_{s,\tau}$ if it is also satisfies*

$(F_{s,\tau})$ if $\inf F=-\infty$ and $x,y,z\in(0,\infty)$ are such that $\tau+F(sx)\leq F(y)$ and $\tau+F(sy)\leq F(z)$ then

$$\tau+F\left(s^2x\right)\leq F(sy),$$

where $\mathbb{F}^*$ is the set of all functions $F:(0,\infty)\to\mathbb{R}$ satisfying the conditions (F_1) and (F_3).

Next, the authors in [11] introduced the notion of F-weak contraction of Hardy-Rogers-type in the setting of b-metric spaces as follows :

Definition 2.8 ***(See** [11, Definition 5.1])* *Let (X,σ) be a b-metric space with constant $s \geq 1$, $a,b,c,e,f \geq 0$ real numbers and $T : X \to X$ an operator. If there exist $\tau > 0$ and $F \in \mathscr{F}_{s,\tau}$ such that for all $x,y \in X$ the inequality $\sigma(Tx,Ty) > 0$ implies*

$$\tau + F(s\sigma(Tx,Ty)) \leq F(A_T^{\sigma}(x,y)),$$

where

$$A_T^{\sigma}(x,y) = a\sigma(x,y) + b\sigma(x,Tx) + c\sigma(y,Ty) + e\sigma(x,Ty) + f\sigma(y,Tx),$$

then T is called an F-weak contraction of Hardy-Rogers-type.

In [11], Lukács and Kajántó proved the fixed point result below.

Theorem 2.9 ***(See** [11, Theorem 5.2])* *Suppose that (X,σ) is a b-metric space with constant $s \geq 1$ and $T : X \to X$ is an F-weak contraction of Hardy-Rogers-type. If either $a+b+c+(s+1)e < 1$ or $a+b+c+(s+1)f < 1$ holds, then every $x_0 \in X$, the sequence $x_{n+1} = Tx_n$ converges to a fixed point of T. Moreover, if $a+e+f < s$ holds as well, then T has exactly one fixed point.*

2.3. Generalized F-weak contraction of Hardy-Rogers-type

In this subsection, we do several improvements in Theorem 2.9. For the sake of readability, we keep some notations used in [11]. Throughout this subsection, (X,σ) represents a b-metric space with constant $s \geq 1$. We recall again for all $x,y \in X$

$$A_T^{\sigma}(x,y) = a\sigma(x,y) + b\sigma(x,Tx) + c\sigma(y,Ty) + e\sigma(x,Ty) + f\sigma(y,Tx),$$

where a,b,c,e,f are nonnegative real numbers. we introduced the following definition.

Definition 2.10 *Let (X,σ) be a b-metric space with constant $s \geq 1$. A mapping $T : X \to X$ is said to be a generalized F-weak contraction of Hardy-Rogers-type if there exist a nondecreasing function $F : (0,\infty) \to \mathbb{R}$ and $\tau \in \mathscr{S}_1$ such that for all $x,y \in X$,*

$$\sigma(Tx,Ty) > 0 \Rightarrow \tau(\sigma(x,y)) + F(s\sigma(Tx,Ty)) \leq F(A_T^{\sigma}(x,y)). \tag{2}$$

Where $\mathscr{S}_1$ is the set of all functions $\tau : (0,\infty) \to (0,\infty)$ which satisfy the following condition :

$$\liminf_{t\to\eta^+} \tau(t) > 0, \quad \textit{for all } \eta > 0. \tag{A_1}$$

Before stating our main results, we proved the following useful lemma.

Lemma 2.11 *Let $\kappa \geq 1$ be a given real number. Let $\{t_n\} \subset (0,\infty)$ be a sequence and let $\phi, \psi : (0,\infty) \to \mathbb{R}$ be two functions satisfying the following conditions :*

- (i) *$\psi(\kappa t_n) \leq \phi(t_{n-1})$, for all $n \in \mathbb{N}$;*
- (ii) *ψ is nondecreasing;*
- (iii) *$\phi(t) < \psi(t)$, for all $t > 0$;*
- (iv) *$\limsup_{t\to\eta^+} \phi(t) < \psi(\eta^+)$, for all $\eta > 0$.*

Then $\lim_{n\to\infty} t_n = 0$.

Also, we proved the following proposition which plays an important role in the proofs of our results.

Proposition 2.12 *Let (X,σ) be a b-metric space with constant $s \geq 1$ and let λ be a given real number such that $1 \leq \lambda \leq s$. Let $T : X \to X$ be a mapping and $\{x_n\}$ the Picard sequence of T based on an arbitrary $x_0 \in X$. Assume that there exist a nondecreasing function F and $\tau \in \mathscr{S}_1$ such that for all $z \in X$ with $Tz \neq T^2 z$,*

$$\begin{aligned} &\tau(\sigma(z,Tz)) + F\left(\lambda\sigma\left(Tz,T^2z\right)\right) \\ &\leq F\left((d_1+d_2)\sigma(z,Tz) + d_3\sigma\left(Tz,T^2z\right) + d_4\sigma\left(z,T^2z\right)\right), \end{aligned} \tag{P}$$

where d_1,d_2,d_3,d_4 are nonnegative real numbers satisfying

$$d_1+d_2+d_3+2d_4 s = \frac{\lambda}{s} \text{ and } d_3 \neq \frac{\lambda}{s}. \tag{D}$$

Then $\lim_{n\to\infty} \sigma(x_n,x_{n+1}) = 0$.

Now, we stated and proved the following theorem.

Theorem 2.13 *Let (X,σ) be a complete b-metric space with constant $s \geq 1$ and $T : X \to X$ a generalized F-weak contraction of Hardy-Rogers-type. Suppose that either $\left(\mathscr{A}_s^1\right)$ or $\left(\mathscr{A}_s^2\right)$ holds, where*

$\left(\mathscr{A}_s^1\right)$ $a+b+c+(s+1)e < 1$,

$\left(\mathscr{A}_s^2\right)$ $a+b+c+(s+1)f < 1$.

Furthermore, we assume that $sa+s^2(e+f) < 1$. Then T has a unique fixed point x^* and for every $x_0 \in X$ the sequence $\{T^n x_0\}_{n\in\mathbb{N}}$ converges to x^*.

Remark 2.14 *Compared with Theorem 2.9, it is clear that Theorem 2.13 gives some improvements. Actually, τ is taken as a function in Theorem 2.13. Moreover, Theorem 2.13 shows that both conditions (F_3) and $(F_{s,\tau})$ from Theorem 2.9 are dropped and replaced by the condition that $sa+s^2(e+f) < 1$. This latter condition is quite simple and ensures simultaneously, with the remaining common hypotheses of Theorem 2.9 and Theorem 2.13, the existence and uniqueness of the fixed point. However, Theorem 2.13 does not cover totally Theorem 2.9, since the condition that $a+e+f < s$ (in Theorem 2.9) which is only used in the uniqueness part is slightly weaker than the condition that $sa+s^2(e+f) < 1$. Besides, the strictness of the monotonicity of F is not necessary.*

In what follows, we presented another proof of Theorem 1-(a) of Hardy-Rogers [7].

Corollary 2.15 ***(See** [7, Theorem 1-(a)]**)** Let (X,d) be a complete metric space and T a self-mapping on X satisfying for all $x,y \in X$,*

$$d(Tx,Ty) \leq \theta_1 d(x,y) + \theta_2 d(x,Tx) + \theta_3 d(y,Ty) + \theta_4 d(x,Ty) + \theta_5 d(y,Tx), \tag{3}$$

where $\theta_i, i = 1,\ldots,5$ are nonnegative numbers such that $\theta = \sum_{i=1}^{5} \theta_i < 1$. Then, T has a unique fixed point.

Example 2.16 *Let $X = [0,5]$ be equipped with the euclidean distance $d = |.|$ and $T : X \to X$ a mapping defined by*

$$Tx = \begin{cases} 5, & \text{if } x \in \,]0,5], \\ \frac{9}{2}, & \text{if } x = 0. \end{cases}$$

Obviously, we get

$$d(Tx,Ty)=\frac{1}{2}>0\Leftrightarrow\left[(x\in\,]0,5]\wedge y=0)\vee(y\in\,]0,5]\wedge x=0)\right]. \tag{4}$$

Let $x,y\in X$ and denote

$$\mathfrak{D}(x,y)=\frac{1}{8}d(x,y)+\frac{1}{4}d(x,Tx)+\frac{1}{4}d(y,Ty)+\frac{1}{16}\left(d(x,Ty)+d(y,Tx)\right).$$

Next, in each of the above cases (4)*, we obtain*

$$\mathfrak{D}(x,y)\geq\frac{1}{4}d(0,T0)=\frac{9}{8}.$$

On the other hand, by using $h+\frac{1}{h}\geq 2$, $\forall h>0$, we get

$$\begin{aligned}\frac{22}{9}-\frac{1}{(d(Tx,Ty))^2+1+(-1)^q}&\leq\frac{22}{9}-\frac{4}{9}=2\\&\leq\mathfrak{D}(x,y)+\frac{1}{\mathfrak{D}(x,y)},\end{aligned}$$

for all $x,y\in X$ with $Tx\neq Ty$ and $q\in\mathbb{N}_0$.

As $d(Tx,Ty)<1$ and $\mathfrak{D}(x,y)>1$, then by choosing $\tau=\frac{22}{9}$ and $F:(0,\infty)\to\mathbb{R}$ defined by

$$F(t)=\begin{cases}-\dfrac{1}{t^2+1+(-1)^q}, & \text{if } 0<t\leq 1,\quad q\in\mathbb{N}_0,\\[2ex] t+\dfrac{1}{t}, & \text{if } t>1,\end{cases}$$

it is easy to see that all the conditions of theorem 2.13 are fulfilled for $a=\frac{1}{8}$, $b=c=\frac{1}{4}$ and $e=f=\frac{1}{16}$ and $\tau=\frac{22}{9}$. Consequently, T has a unique fixed point x^* (which is 5).

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