ON THE GEOMETRIC-WEIGHTED-VARIABLE HARDY SPACES ON LIPSCHITZ DOMAINS

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ABSTRACT

In this paper, we introduce and explore the geometric-weighted Hardy spaces with variable exponent on bounded Lipschitz domain $\mathbb{D}$ of $\mathbb{R}^n$.

1. INTRODUCTION

The variable Lebesgue spaces is a well known generalization for the classical Lebesgue spaces and it has become an interesting field for scientists in recent decades due to its wide uses and applications in real world phenomena. The variable Lebesgue spaces $L^{p(\cdot)}(\Omega)$ was initiated first by Orlicz [8], then studied and investigated by many authors see for example [7, 2, 3, 4].

The classical Hardy space on $\mathbb{R}^n$ was introduced and developed by Stein and Weiss [15]. This kind of space was also extended to the variable setting. In particular, Nakai and Sawano [12] introduced and investigated the Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ with variable exponent, where they established the atomic decomposition of this space. As in the results established in [12], Liu [9] extended the results obtained by Miyachi [11] from the real variable Hardy space on domains to the Hardy spaces with variable exponents. More precesily, Liu [9] obtained the atomic decomposition of the variable Hardy spaces on domains and as an application, he studied the geometric Hardy spaces with variable exponent. On the other hand, the weighted variable Hardy spaces are considered as an extension for the variable Hardy spaces. K-P. Ho [5] introduced and studied the variable weighted Hardy spaces $H_w^{p(\cdot)}(\mathbb{R}^n)$, where he introduced a general class of weights compared to the ordinary Mukenhoupt class of weights. More recently, O. Melkemi et al. [10] extended the results obtained by K-P. Ho [5], where the authors introduced and investigated the atomic characterization of the weighted variable Hardy spaces on general domains Ω of $\mathbb{R}^n$. For more information and results concerning the variable Hardy spaces we refer the reader to [16, 14, 13, 6] and the references therein. Motivated by the recent works in [10] and [9], our main goal is to investigate the geometric-weighted-variable Hardy spaces $H_{w,\mathrm{r}}^{p(\cdot)}(\mathbb{D})$ on Lipschitz domain $\mathbb{D}$ of $\mathbb{R}^n$. We recall that the spaces $H_{w,\mathrm{r}}^{p(\cdot)}(\mathbb{D})$ are defined by means restricting arbitrary elements of $H_w^{p(\cdot)}(\mathbb{R}^n)$ to a bounded domain $\mathbb{D}$. By applying the atomic characterization of the space $H_w^{p(\cdot)}(\mathbb{D})$ obtained in [10] for a general domains of $\mathbb{R}^n$, the reflection technique for Lipschitz domains and borrowing some ideas from [1], we prove the following identity $H_w^{p(\cdot)}(\mathbb{D}) = H_{w,r}^{p(\cdot)}(\mathbb{D})$

(see Theorem 3 below). The rest of this paper is arranged as follows. In the next section, we give some basic definitions and ingredients. In the last Section, we state and prove the main result obtained in this article.

As usual, throughout this paper, C stands for a positive constant which may be different from line to line, $A \lesssim B$ means that there exists a positive constant C such that $A \leq CB$, the symbol $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

2. PRELIMINARIES

A measurable function $p(\cdot) : \mathbb{D} \to (0, \infty)$ is called a variable exponent. For a variable exponent $p(\cdot)$, define

$$p_- = \operatorname{ess\,inf}_{x \in \mathbb{D}} p(x) \quad \text{and} \quad p_+ = \operatorname{ess\,sup}_{x \in \mathbb{D}} p(x),$$

and let us denote by $\mathscr{P}(\mathbb{D})$ the collection of all variable exponents such that $0 < p_- \leq p_+ < \infty$.

Let $p(\cdot) \in \mathscr{P}(\mathbb{D})$. The Lebesgue space with variable exponents $L^{p(\cdot)}(\mathbb{D})$ consists of all measurable functions $f : \mathbb{D} \to \mathbb{R}$ such that $\int_{\mathbb{D}} |f(x)|^{p(x)} dx < \infty$, equipped with the Luxemburg quasinorm

$$\|f\|_{L^{p(\cdot)}(\mathbb{D})} = \inf\Big\{ \lambda > 0 : \int_{\mathbb{D}} \left[\frac{|f(x)|}{\lambda} \right]^{p(x)} dx \leq 1 \Big\}. \tag{2.1}$$

In the following lemma, we collect some useful properties of the variable exponent Lebesgue spaces. For the proofs and more details about these spaces, we refer to [3].

Lemma 1 *Let $p(\cdot) \in \mathscr{P}(\mathbb{D})$ and $f, g \in L^{p(\cdot)}(\mathbb{D})$.*

(1) For $\lambda \in \mathbb{C}$, we have $\|\lambda f\|_{L^{p(\cdot)}(\mathbb{D})} = |\lambda| \|f\|_{L^{p(\cdot)}(\mathbb{D})}$.

(2) For any $s \in (0, \infty)$, we have $\||f|^s\|_{L^{p(\cdot)}(\mathbb{D})} = \|f\|^s_{L^{sp(\cdot)}(\mathbb{D})}$.

(3) $\|f + g\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{D})} \leq \|f\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{D})} + \|g\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{D})}$ where $\underline{p} = \min\{p_-, 1\}$.

We recall that for any $f \in L^1_{\text{loc}}(\mathbb{D})$, the Hardy-Littlewood maximal operator M is defined for all $x \in \mathbb{D}$ by setting,

$$M(f)(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| \, dy,$$

where the supremum is taken over all balls B of $\mathbb{D}$ containing x.

Next, we recall the definition and some properties of the weighted variable Lebesgue spaces. Let $w : \mathbb{D} \to (0, \infty)$ be a locally integrable function. The weighted variable exponent Lebesgue space $L^{p(\cdot)}_w(\mathbb{D})$ is defined as the space of all measurable functions $f : \mathbb{D} \to \mathbb{C}$ such that

$$\|f\|_{L^{p(\cdot)}_w(\mathbb{D})} = \|fw\|_{L^{p(\cdot)}(\mathbb{D})} < \infty.$$

The weights used in this paper belongs to the following class.

Definition 1 *Let $p(\cdot) : \mathbb{D} \to (0, \infty)$ be a measurable function such that $0 < p_- \leq p_+ < \infty$ and $w : \mathbb{D} \to (0, \infty)$ be a Lebesgue measurable function. We denote by $W_{p(\cdot)}(\mathbb{D})$ the set of all Lebesgue measurable functions w such that*

(1) $\|\chi_B\|_{L^{p(\cdot)/\underline{p}}_{w^{\underline{p}}}(\mathbb{D})} < \infty$ and $\|\chi_B\|_{L^{(\frac{p(\cdot)}{\underline{p}})'}_{w^{-\underline{p}}}(\mathbb{D})} < \infty$, for any ball B of $\mathbb{D}$;

(2) there exists $k > 1$ and $s > 1$ such that the Hardy-Littlewood maximal operator is bounded on $L^{(sp(\cdot))'/k}_{w^{-k/s}}(\mathbb{D})$.

For any $w \in W_{p(\cdot)}(\mathbb{D})$, set

$$s_w = \inf\{s \geq 1 : M \text{ is bounded on } L^{(sp(\cdot))'}_{w^{-1/s}}(\mathbb{D})\},$$

$$\mathbb{S}_w = \{s \geq 1 : M \text{ is bounded on } L^{(sp(\cdot))'/k}_{w^{-k/s}}(\mathbb{D}) \text{ for some } k > 1\}$$

and

$$k^s_w = \sup\{k \geq 1 : M \text{ is bounded on } L^{(sp(\cdot))'/k}_{w^{-k/s}}(\mathbb{R}^n)\}. \tag{2.2}$$

The following theorem gives the Fefferman-Stein vector valued maximal inequalities on $L^{p(\cdot)}_w(\mathbb{D})$. For the proof, we refer to [5].

Theorem 2 *Let $p(\cdot) : \mathbb{D} \to (0,\infty)$ be a measurable function with $0 < p_- \leq p_+ < \infty$ and $q \in (1,\infty)$. If $w \in W_{p(\cdot)}(\mathbb{D})$, then, for any $r > s_w$, we have*

$$\left\| \left(\sum_{i\in\mathbb{N}} (Mf_i)^q \right)^{1/q} \right\|_{L^{rp(\cdot)}_{w^{1/r}}(\mathbb{D})} \leq C \left\| \left(\sum_{i\in\mathbb{N}} |f_i|^q \right)^{1/q} \right\|_{L^{rp(\cdot)}_{w^{1/r}}(\mathbb{D})}.$$

Let $\phi \in \mathscr{D}(B(\mathbf{0}_n,1))$ such that $\int_{\mathbb{R}^n} \phi(x)dx = 1$. For any $t \in (0,\infty)$ and $x \in \mathbb{D}$, we set $\phi_t(x) = t^{-n}\phi(t^{-1}x)$. For any $f \in \mathscr{D}'(\mathbb{D})$, the radial maximal function $\mathscr{M}^+_{\phi,\Omega}(f)$ is defined for any $x \in \mathbb{D}$ by

$$\mathscr{M}^+_{\phi,\Omega}(f)(x) := \sup_{t\in(0,\mathrm{dist}(x,\mathbb{D}^c))} |\langle f, \phi_t(x-\cdot)\rangle|, \tag{2.3}$$

where $\mathbb{D}^c$ denotes the complementary set of $\mathbb{D}$ in $\mathbb{R}^n$, $\mathrm{dist}(x,\mathbb{D}^c) := \inf\{|x-y| \ : \ y \in \mathbb{D}^c\}$ and $\langle\cdot,\cdot\rangle$ denotes the duality between $\mathscr{D}'(\mathbb{D})$ and $\mathscr{D}(\mathbb{D})$.

Now, we introduce weighted variable Hardy space on domains and we establish its atomic and maximal function characterizations.

Definition 2 *Let $\mathbb{D}$ be an open set of $\mathbb{R}^n$ and $p(\cdot) \in \mathscr{P}(\mathbb{D})$. Then, the weighted variable Hardy space $H^{p(\cdot)}_w(\mathbb{D})$ is defined to be the set of all $f \in \mathscr{D}'(\mathbb{D})$ such that $\mathscr{M}^+_{\phi,\Omega}(f) \in L^{p(\cdot)}_w(\mathbb{D})$, where $\mathscr{M}^+_{\phi,\Omega}$ is as in* (2.3)*, equipped with the quasi-norm*

$$\|f\|_{H^{p(\cdot)}_w(\mathbb{D})} = \|\mathscr{M}^+_{\phi,\Omega}(f)\|_{L^{p(\cdot)}_w(\mathbb{D})}.$$

Next, we give the definition of $(p(\cdot),r,w)$-atoms in $\mathbb{D}$.

Definition 3 *Let $\mathbb{D}$ be an open set of $\mathbb{R}^n$, $p(\cdot) \in \mathscr{P}(\mathbb{D})$, $w : \mathbb{D} \to (0,\infty)$, $q \in (1,\infty]$ and*

$$d_w = n(s_w - 1). \tag{2.4}$$

1. *A cube $Q \subset \mathbb{R}^n$ is called of type (a) if $4Q \subset \mathbb{D}$ and $\widetilde{Q} \subset \mathbb{R}^n$ is called of type (b) if $2\widetilde{Q} \cap \mathbb{D}^c = \varnothing$ and $4\widetilde{Q} \cap \mathbb{D}^c \neq \varnothing$.*
2. *A measurable function a on $\mathbb{D}$ is called a type (a) $(p(\cdot),q,w)_{\mathbb{D}}-$atom if there exists a cube Q of type (a) such that*
   - *(1) supp $a \subset Q$;*
   - *(2) $\|a\|_{L^q(\mathbb{D})} \leq \frac{|Q|^{1/q}}{\|\chi_Q\|_{L^{p(\cdot)}_w(\mathbb{D})}}$;*
   - *(3) there exist $s \geq d_w$ such that, $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ for all $\alpha \in \mathbb{Z}^n_+$ with $|\alpha| \leq s$.*
3. *A measurable function b on $\mathbb{D}$ is called a type (b) $(p(\cdot),q,w)_{\mathbb{D}}-$atom if there exists a cube $\widetilde{Q}$ of type (b) such that*

(1) $\operatorname{supp} b \subset \widetilde{Q}$;

(2) $\|b\|_{L^q(\mathbb{D})} \leq \frac{|\widetilde{Q}|^{1/q}}{\|\chi_{\widetilde{Q}}\|_{L_w^{p(\cdot)}(\mathbb{D})}}$;

Let $p(\cdot) \in \mathscr{P}(\mathbb{D})$, $\{\lambda_i\}_{i\in\mathbb{N}}$ be a sequence of numbers in $\mathbb{C}$, $\{Q_i\}_{i\in\mathbb{N}}$ be a cube sequence of the supports of type (a) $(p(\cdot), r, w)_{\mathbb{D}}-$ atoms, $\{\kappa_i\}_{i\in\mathbb{N}}$ be a sequence of numbers in $\mathbb{C}$, $\{\widetilde{Q}_i\}_{i\in\mathbb{N}}$ be a cube sequence of the supports of type (b) $(p(\cdot), r, w)_{\mathbb{D}}-$ atoms. Define

$$\mathscr{A}(\{\lambda_i\}_{i\in\mathbb{N}}, \{Q_i\}_{i\in\mathbb{N}}) := \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i|\chi_{Q_i}}{\|\chi_{Q_i}\|_{L_w^{p(\cdot)}(\mathbb{D})}} \right]^{\theta} \right\}^{\frac{1}{\theta}} \right\|_{L_w^{p(\cdot)}(\mathbb{D})},$$

and

$$\mathscr{B}(\{\kappa_i\}_{i\in\mathbb{N}}, \{\widetilde{Q}_i\}_{i\in\mathbb{N}}) := \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\kappa_i|\chi_{\widetilde{Q}_i}}{\|\chi_{\widetilde{Q}_i}\|_{L_w^{p(\cdot)}(\mathbb{D})}} \right]^{\theta} \right\}^{\frac{1}{\theta}} \right\|_{L_w^{p(\cdot)}(\mathbb{D})},$$

3. MAIN RESULT

In this section, we present our main result. We start by introducing the geometric-weighted Hardy space with variable exponent on a proper open subset $\mathbb{D}$ of $\mathbb{R}^n$.

Definition 4 *Let $\mathbb{D} \subset \mathbb{R}^n$ be a proper open subset and $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$. Denote by $H_w^{p(\cdot)}(\mathbb{R}^n)$ the weighted variable Hardy spaces on $\mathbb{R}^n$. Then, it is said that a distribution $f \in \mathscr{D}'(\mathbb{D})$ belongs to $H_{w,r}^{p(\cdot)}(\mathbb{D})$, if f is the restrection to $\mathbb{D}$ of $\mathrm{F} \in H_w^{p(\cdot)}(\mathbb{R}^n)$. Namely, we have*

$$H_{w,r}^{p(\cdot)}(\mathbb{D}) := \Big\{ f \in \mathscr{D}'(\mathbb{D}) : \text{ there exists } \mathrm{F} \in H_w^{p(\cdot)}(\mathbb{R}^n) : \text{ such that } \mathrm{F}|_{\mathbb{D}} \equiv f \Big\}.$$

The definition of $(p(\cdot), q, w)-$atoms on $\mathbb{R}^n$ is given as follows

Definition 5 *Let $p(\cdot) \in \mathscr{P}(\mathbb{D})$, $w : \mathbb{D} \to (0, \infty)$, $q \in (1, \infty]$ and*

$$d_w = n(s_w - 1).$$

A measurable function a on $\mathbb{R}^n$ is called $(p(\cdot), q, w)_{\mathbb{D}}-$atom if there exists a cube Q such that

(1) $\operatorname{supp} a \subset Q$;

(2) $\|a\|_{L^q(\mathbb{R}^n)} \leq \frac{|Q|^{1/q}}{\|\chi_Q\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}}$;

(3) there exist $s \geq d_w$ such that, $\int_{\mathbb{R}^n} a(x) x^{\alpha} dx = 0$ for all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \leq s$.

Definition 6 *For sequences of $\{\lambda_i\}_{i\in\mathbb{N}} \subset \mathbb{C}$ and cubes $\{Q_j\}_{i\in\mathbb{N}}$, define that*

$$\mathscr{A}'(\{\lambda_i\}_{i\in\mathbb{N}}, \{Q_i\}_{i\in\mathbb{N}}) := \left\| \left\{ \sum_{i\in\mathbb{N}} \left[\frac{|\lambda_i|\chi_{Q_i}}{\|\chi_{Q_i}\|_{L_w^{p(\cdot)}(\mathbb{D})}} \right]^{\theta} \right\}^{\frac{1}{\theta}} \right\|_{L_{w^{\theta}}^{p(\cdot)/\theta}(\mathbb{D})},$$

where $\theta \in \mathbb{S}_w$.

The result of this section is given below

Theorem 3 *Let $\mathbb{D} \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $p(\cdot) \in C^{log}(\mathbb{R}^n)$ and $w \in W_{p(\cdot)}(\mathbb{R}^n)$ with $\frac{n}{1+n} \leq p_- \leq p_+ \leq 1$. Then we have the following identity :*

$$H_w^{p(\cdot)}(\mathbb{D}) = H_{w,r}^{p(\cdot)}(\mathbb{D}),$$

with equivalent quasi-norms.

Proof of Theorem 3.

We first show that, for any $f \in H^{p(\cdot)}_{w,r}(\mathbb{D})$ we have $\|f\|_{H^{p(\cdot)}_{w}(\mathbb{D})} \lesssim \|f\|_{H^{p(\cdot)}_{w,r}(\mathbb{D})}$. Let $f \in H^{p(\cdot)}_{w,r}(\mathbb{D})$ then by definition there exists $\mathrm{F} \in H^{p(\cdot)}_{w}(\mathbb{R}^n)$ such that $\mathrm{F}|_{\mathbb{D}} \equiv f$. According to [5, Theorem 5.3] and (6), we infer that there exist a sequence $\{\lambda_Q\}_Q \subset \mathbb{C}$, a sequence $\{a_Q\}_Q$ of $(p(\cdot),q)$-atoms and $\mathscr{A}'(\{\lambda_Q\}_Q, \{Q\}_Q) < \infty$ such that $\mathrm{F} = \sum_Q \lambda_Q a_Q$ in $\mathscr{S}'(\mathbb{R}^n)$ and

$$\|\mathrm{F}\|_{H^{p(\cdot),q}_{\mathrm{atom},w}} \sim \mathscr{A}'(\{\lambda_Q\}_Q, \{Q\}_Q), \tag{3.1}$$

where, those atoms a_Q supported in cubes $Q \in \mathbb{D}$ with $(4Q) \cap \mathbb{D}^c = \emptyset$, are treated as type $(a)(p(\cdot),q)_{\mathbb{D}}-$atoms and we have

$$\mathscr{A}(\{\lambda_Q\}_{Q_1}, \{Q\}_{Q_1}) = \mathscr{A}'(\{\lambda_Q\}_{Q_1}, \{Q\}_{Q_1}) < \infty, \tag{3.2}$$

where $Q_1 = \{Q \subset \mathbb{D} : (4Q) \cap \mathbb{D}^c = \emptyset\}$.

We treat as type $(b)(p(\cdot),q)_{\mathbb{D}}$-atoms and we have

$$\mathscr{B}(\{\lambda_Q\}_{Q_2}, \{Q\}_{Q_2}) = \mathscr{A}'(\{\lambda_Q\}_{Q_2}, \{Q\}_{Q_2}) < \infty, \tag{3.3}$$

where $Q_2 = \{Q \subset \mathbb{D} : (4Q) \cap \mathbb{D}^c \neq \emptyset \text{ and } (2Q) \cap \mathbb{D}^c = \emptyset\}$. For those a_Q atoms we can decompose them into type $(b)(p(\cdot),\infty)_{\mathbb{D}}$-atoms via the Whitney decomposition. To do so, we consider atoms in the decomposition of F that intersect $\partial\mathbb{D}$. When we restrict the atom to $\mathbb{D}$, we will lose part of its support. Without lose of generality we consider an atom A fulfills $\|A\|_{L^q(\mathbb{R}^n)} \leq 1$, with $q \in (1,\infty]$, and its supported on the cube

$$Q = \{(x', x_n) : |x_j| \leq 1/2 \text{ when } j = 1, \cdots, n-1, 0 < x_n \leq \alpha\},$$

for some $\alpha \in (0,1)$. From the Whitney decomposition on $\mathbb{D}$ of Q with respect to $\partial\mathbb{D}$, we find that, the cube Q is decomposed into a family of sub-cubes $\{Q^k_j\}$ of distance 2^{-k} from $\partial\mathbb{D}$ and there are $c_k \sim 2^{(n-1)k}$ of them. We can observe that j varies from 1 to c_k. From the Whitney decomposition, it follows that each cube Q^k_j is a type (b) cube and

$$A(x', x_n) = \sum_{k=1}^{\infty} \sum_{j=1}^{c_k} \chi_{Q^k_j} A = \sum_{k=1}^{\infty} \sum_{j=1}^{c_k} \lambda_{Q^k_j} a_{Q^k_j},$$

where, for $j = 1, \cdots, c_k$ and $k \in \mathbb{N}$,

$$\lambda_{Q^k_j} := \|\chi_{Q^k_j} A\|_{L^\infty(\mathbb{D})} \|\chi_{Q^k_j}\|_{L^{p(\cdot)}_w(\mathbb{D})}$$

and

$$a_{Q^k_j} := \frac{\chi_{Q^k_j} A}{\|\chi_{Q^k_j} A\|_{L^\infty(\mathbb{D})} \|\chi_{Q^k_j}\|_{L^{p(\cdot)}_w(\mathbb{D})}}.$$

From the fact that $a_{Q^k_j}$ is supported in type (b) cube Q^k_j, we obtain

$$\|a_{Q^k_j}\|_{L^\infty(\mathbb{D})} \leq \frac{\chi_{Q^k_j} A}{\|\chi_{Q^k_j} A\|_{L^\infty(\mathbb{D})} \|\chi_{Q^k_j}\|_{L^{p(\cdot)}_w(\mathbb{D})}} \leq \frac{1}{\|\chi_{Q^k_j}\|_{L^{p(\cdot)}_w(\mathbb{D})}}.$$

Hence, each $a_{Q^k_j}$ is a type $(b)(p(\cdot),\infty)_{\mathbb{D}}-$atom, Moreover, by combining the Hölder inequality, $\|A\|_{L^q(\mathbb{R}^n)} \leq 1, \sum_{k=1}^{\infty} \sum_{j=1}^{c_k} |Q^k_j| = |Q|$ and $\|\chi_Q\|_{L^{p(\cdot)}_w(\mathbb{R}^n)} \lesssim |Q|^{1/p_+}$, we find out that

$$\begin{aligned}
\mathscr{B}(\{\lambda_{Q_j^k}\}_{j,k},\{Q_j^k\}_{j,k}) &= \mathscr{A}'(\{\lambda_{Q_j^k}\}_{j,k},\{Q_j^k\}_{j,k}) \\
&\leq \left\| \Big(\sum_{k=1}^{\infty}\sum_{j=1}^{c_k} \|\chi_{Q_j^k} A\|_{L^\infty(\mathbb{D})} \Big) \Big(\sum_{k=1}^{\infty}\sum_{j=1}^{c_k} \chi_{Q_j^k} \Big)^{\frac{1-\theta}{\theta}} \right\|_{L_{w^\theta}^{p(\cdot)/\theta}(\mathbb{R}^n)} \qquad (3.4)\\
&\leq \|A\|_{L^\infty(\mathbb{R}^n)} \|\chi_Q\|_{L_w^{p(\cdot)}(\mathbb{R}^n)} \\
&\lesssim |Q|^{1/p_+} < \infty,
\end{aligned}$$

which means that the atomic decomposition converges in $H_w^{p(\cdot)}(\mathbb{R}^n)$. Furthermore, in view of [10, Theorem 3.7], (3.2), (3.3) and (3.4), we get

$$\begin{aligned}
\|f\|_{H^{p(\cdot)}(\mathbb{D})} &\sim \|f\|_{H_{q,\mathrm{atom}}^{p(\cdot),q}(\mathbb{D})} \\
&\leq \mathscr{A}(\{\lambda_Q\}_{Q1},\{Q\}_{Q_1}) + \mathscr{B}(\{\lambda_Q\}_{Q_2},\{Q\}_{Q_2}) + \mathscr{B}(\{\lambda_{Q_j^k}\}_{j,k},\{Q_j^k\}_{j,k}) \\
&\sim \mathscr{A}'(\{\lambda_Q\}_{Q1},\{Q\}_{Q_1}) + \mathscr{A}'(\{\lambda_Q\}_{Q_2},\{Q\}_{Q_2}) + \mathscr{A}'(\{\lambda_{Q_j^k}\}_{j,k},\{Q_j^k\}_{j,k}) < \infty,
\end{aligned}$$

from this, (3.1), the definition of the variable geometric-weighted-Hardy space and [5, Theorem 5.3], we deduce that

$$\|f\|_{H_w^{p(\cdot)}(\mathbb{D})} \leq \|\mathrm{F}\|_{H_{w,\mathrm{atom}}^{p(\cdot),q}(\mathbb{R}^n)} \lesssim \|f\|_{H_{r,w}^{p(\cdot)}(\mathbb{D})} < \infty,$$

which implies that $f \in H^{p(\cdot)}(\mathbb{D})$.

Let us now turn out to show that, for $f \in H_w^{p(\cdot)}(\mathbb{D})$ then we have, $\|f\|_{H_{w,r}^{p(\cdot)}(\mathbb{D})} \leq \|f\|_{H_w^{p(\cdot)}(\mathbb{D})}$. Let $f \in H_w^{p(\cdot)}(\mathbb{D})$, then by [10, Theorem 3.7] and Definition 5, we know that there exist two sequences $\lambda_{j\,j\in\mathbb{N}} \subset \mathbb{C}$ and $\kappa_{j\,j\in\mathbb{N}} \subset \mathbb{C}$, a sequence $a_{j\,j\in\mathbb{N}}$ of type $(a)(p(\cdot),q)_{\mathbb{D}}$atoms and a sequence $b_{j\,j\in\mathbb{N}}$ of type $(b)(p(\cdot),q)_{\mathbb{D}}$atoms such that

$$f = \sum_{j\in\mathbb{N}} \lambda_j a_j + \sum_{j\in\mathbb{N}} \kappa_j b_j \quad \text{in } \mathscr{D}'(\mathbb{D}), \tag{3.5}$$

and

$$\mathscr{A}(\{\lambda_j\}_{j\in\mathbb{N}},\{Q_j\}_{j\in\mathbb{N}}) + \mathscr{B}(\{\kappa_j\}_{j\in\mathbb{N}},\{\tilde{Q}_j\}_{j\in\mathbb{N}}) < \infty.$$

The type $(a)(p(\cdot),q)_{\mathbb{D}}$atoms a_j in (3.5) are already $(p(\cdot),q)$atoms and we have

$$\mathscr{A}(\{\lambda_j\}_{j\in\mathbb{N}},\{Q_j\}_{j\in\mathbb{N}}) = \mathscr{A}'(\{\lambda_j\}_{j\in\mathbb{N}},\{Q_j\}_{j\in\mathbb{N}}) < \infty.$$

While, we treat the $(b)(p(\cdot),q)_{\mathbb{D}}$atoms b_j in (3.5) in two different cases.

Case 1. If b_j is of type $(b)(p(\cdot),q)_{\mathbb{D}}$ atoms as in (3.5) and is supported on a type (b) cube Q with $\ell(Q) \leq \varepsilon$ for some ε sufficiently small $\varepsilon > 0$, then we find a cube $\tilde{Q} \subset (\partial\bar{\mathbb{D}})^c$ which has the same size of Q. Now, we set the extension $(b_j)_*$ of function b_j as follows :

$$(b_j)_*(x) := \begin{cases} b_j(x) & \text{for} \quad x \in Q; \\ -\frac{1}{|Q|}\int_Q b_j(y)dy & \text{for} \quad x \in \tilde{Q}, \end{cases}$$

from this, we infer that the function $(b_j)_*$ is supported on $Q \cup \tilde{Q}$. As the distance between Q and $\tilde{Q}$ to $\partial\mathbb{D}$ is comparable to $\ell(Q)$, we may find another cube denoted $\hat{Q}$ such that $Q \cup \tilde{Q}$ is a subset of $\hat{Q}$ and $|Q| \leq |\hat{Q}| \lesssim |Q|$. From this and the Hölder inequality, we obtain

$$\begin{aligned}\|(b_j)_*\|_{L^q(\mathbb{R}^n)} &\leq \|b_j\|_{L^q(\mathbb{R}^n)} + \left\| \left(-\frac{1}{|Q|}\int_Q b_j(y)dy \right)\chi_{\tilde{Q}} \right\|_{L^q(\mathbb{R}^n)} \\ &\leq \|b_j\|_{L^q(\mathbb{R}^n)} + |\tilde{Q}||Q|^{-1}\int_Q b_j(y)dy \\ &\leq \|b_j\|_{L^q(\mathbb{R}^n)} + |\tilde{Q}|^{1/q}|Q|^{-1+1/q'}\|b_j\|_{L^q(\mathbb{R}^n)} \lesssim \frac{|\hat{Q}|^{1/q}}{\|\chi_{\hat{Q}}\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}}\end{aligned}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n}(b_j)_*(x)dx &= \int_Q b_j(x)dx - \int_{\tilde{Q}}\big(\frac{1}{|Q|}\int_Q b_j(y)dy\big)\chi_{\tilde{Q}}(x)dx \\ &= \int_Q b_j(x)dx - |\tilde{Q}||Q|^{-1}\int_Q b_j(x)dx = 0.\end{aligned}$$

Thus, we find out

$$\mathscr{A}'(\{\kappa_Q\}_{Q_3},\{Q\}_{Q_3}) = \mathscr{B}(\{\kappa_Q\}_{Q_3},\{Q\}_{Q_3}) < \infty, \tag{3.6}$$

where $Q_3 := \{Q \subset \mathbb{D} : Q \text{ be a type } (b) \text{ cube with} \ell(Q) \leq \varepsilon\}$.

Case 2. If b_j is a type $(b)(p(\cdot,q)_{\mathbb{D}}$atom in (3.5) and is supported on a type (b) cube Q with $\ell(Q) > \varepsilon$, then from the fact that $\mathbb{D} \subset \mathbb{R}^n$ is a bounded Lipschitz domain, we find out that there exists a cube $\tilde{Q} \subset (\partial\bar{\mathbb{D}})^c$, such that $\ell(\tilde{Q}) = \ell(Q)$ and $\mathrm{dist}(Q,\tilde{Q}) \lesssim \ell(Q)$. Then we can find another cube denoted $\hat{Q}$, such that $Q \cup \tilde{Q} \subset \hat{Q}$ and $\ell(\hat{Q}) = \ell(Q)$. As in **Case 1.** we define the function

$$(b_j)_\sharp(x) := \begin{cases} b_j(x) & \text{for} \quad x \in Q; \\ -\frac{1}{|Q|}\int_Q b_j(y)dy & \text{for} \quad x \in \tilde{Q}, \end{cases}$$

which implies that $(b_j)_\sharp$ is supported on $Q \cup \tilde{Q}$, such that

$$\|(b_j)_\sharp\|_{L^q(\mathbb{R}^n)} \leq \|b_j\|_{L^q(\mathbb{R}^n)} + |Q|^{-1}\|(\int_Q b_j(y)dy)\chi_{\tilde{Q}}\|_{L^q(\mathbb{R}^n)} \lesssim \frac{|Q|^{1/q}}{\|\chi_{\tilde{Q}}\|_{L_w^{p(\cdot)}(\mathbb{R}^n)}}$$

and

$$\begin{aligned}\int_{\mathbb{R}^n}(b_j)_\sharp(x)dx &= \int_Q b_j(x)dx - \int_{\tilde{Q}}\big(\frac{1}{|Q|}\int_Q b_j(y)dy\big)\chi_{\tilde{Q}}(x)dx \\ &= \int_Q b_j(x)dx - |\tilde{Q}||Q|^{-1}\int_Q b_j(x)dx = 0.\end{aligned}$$

Consequently, we obtain

$$\mathscr{A}'(\{\kappa_Q\}_{Q_4},\{Q\}_{Q_4}) = \mathscr{B}(\{\kappa_Q\}_{Q_4},\{Q\}_{Q_4}) < \infty, \tag{3.7}$$

where $Q_4 := \{Q \subset \mathbb{D} : Q \text{ be a type (b) cube with} \ell(Q) > \varepsilon\}$. Thus By Definition of the space $H_{w,r}^{p(\cdot)}(\mathbb{D})$, (3.1), (3.6), (3.7) and [5, Theorem 5.3], we infer that

$$\begin{aligned}\|f\|_{H_{w,r}^{p(\cdot)}(\mathbb{D})} &\leq \|\mathrm{F}\|_{H_w^{p(\cdot)}(\mathbb{R}^n)} \sim \|\mathrm{F}\|_{H_{w,\mathrm{atom}}^{p(\cdot),q}(\mathbb{R}^n)} \\ &\leq \mathscr{A}'(\{\kappa_{\lambda_j}\}_{j\in\mathbb{N}},\{Q_j\}_{j\in\mathbb{N}}) + \mathscr{A}'(\{\kappa_Q\}_{Q_3},\{Q\}_{Q_3}) + \mathscr{A}'(\{\kappa_Q\}_{Q_4},\{Q\}_{Q_4}) < \infty.\end{aligned}$$

In view of [10, Theorem 3.7], we obtain that

$$\|f\|_{H^{p(\cdot)}_{w,r}(\mathbb{D})} \leq \|f\|_{H^{p(\cdot),q}_{w,\mathrm{atom}}(\mathbb{D})} \sim \|f\|_{H^{p(\cdot)}_{w}(\mathbb{D})} < \infty,$$

which is the required result. ■

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