NONLOCAL CONDITIONS FOR FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In this work we use the method of lower and upper solutions to develop an iterative technique, which is not necessarily monotone, and combined with a fixed point theorem to prove the existence of at least one solution of nonlinear fractional differential equations with nonlocal boundary conditions of integral type.

1. INTRODUCTION

In this paper, we consider the following class of fractional differential equations

$$D^{\alpha}u(t)+\ f(t,u)=0,\quad t\in(0,1)\,,\ 1<\alpha\leq 2, \tag{1}$$

with a Newmann condition at the initial point and a nonlocal boundary condition of integral type at the terminal point

$$u'(0)=0,\ \ u(1)=\int_{0}^{1} g\left(u(t)\right)dt. \tag{2}$$

Here D^{α} is the Caputo fractional derivative of order $\alpha\in(1,2]$, $f:[0,1]\times\mathbb{R}\to\mathbb{R}$, and $g:\mathbb{R}\to\mathbb{R}$ satisfy conditions that will be specified later. We use the lower and upper solutions method to develop an iterative method, which is not necessarily monotone (see [**?**],[**?**]) and combined with the Schauder fixed theorem to prove the existence of least one solution for problem $(1)-(2)$.

The rest of this paper is organized as follows. In Section 2 we recall some basic definitions and results that are needed in the rest of the paper. In Section 3, we develop the iterative technique in order to prove our main result concerning the existence of the solution of the problem $(1)-(2)$. Finally, we give an example to illustrate our main result.

2. PRELIMINARIES

In this section, we recall some basic definitions, notations and few results from fractional calculus that we shall use in the remaider of the paper. Let I denote the compact real interval $[0,1]$ and let $C^{n}(I)$, $n\in\mathbb{N}$, is the space of continuous functions $\omega:I\to\mathbb{R}$, such that $\omega^{(k)}\in C^{0}(I)$ $k=0,1,2,...,n$, equipped with the norm

$$\|\omega\|_{C^n}=\sum_{k=0}^{n}\max_{0\leq t\leq 1}\left|\omega^{(k)}(t)\right|.$$

Definition 1 *(see [6]) The Riemann-Liouville fractional primitive of order $\alpha > 0$ of a function $f : (0, \infty) \to \mathbb{R}$ is given by*

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \tag{3}$$

provided that the right-hand side is pointwise defined on $(0, +\infty)$, and where Γ is the gamma function.

For instance, I^{α} exists for all $\alpha > 0$, when $f \in C(I)$. Notice, also, that when $f \in C(I)$, then $I^{\alpha} f \in C(I)$ and moreover $I^{\alpha} f(0) = 0$. The law of composition $I^{\alpha} I^{\beta} = I^{\alpha+\beta}$ holds for all $\alpha, \beta > 0$.

Definition 2 *(see [6]) The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, \infty) \to \mathbb{R}$ is given by*

$$D^{\alpha} f(t) = I^{n-\alpha} f^{(n)}(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau, \tag{4}$$

where $n = [\alpha] + 1$ and $[\alpha]$ is the integer part of α, provided that the right-hand side is pointwise defined on $(0, +\infty)$.

Notice that $D^{\alpha} c = 0$, where c is a real constant.

Remark 1 *It is well known , [6, Lemma 2.22, page 96], that for $\alpha > 0$*

$$I^{\alpha} D^{\alpha} u(t) = u(t) + c_0 + c_1 t + ... + c_{n-1} t^{n-1}, \text{ for all } t \in I,$$

where $n = [\alpha] + 1$, and $c_0, c_1, ..., c_{n-1}$ are real constants.

Lemma 1 *Let $\alpha > 0$. Then the differential equation*

$$D^{\alpha} u(t) = 0$$

has solutions $u(t) = c_0 + c_1 t + ... + c_{n-1} t^{n-1}$, $t \in I$, $c_0, c_1, ..., c_{n-1}$ are real constants and $n = [\alpha] + 1$.

Lemma 2 *Let $\alpha \in (1, 2)$. Then the homogeneous problem*

$$\begin{cases} D^{\alpha} u(t) = 0, & t \in I \\ u'(0) = 0,\ u(1) = 0 \end{cases}$$

has only the trivial solution $u(t) = 0$ for all $t \in I$.

Lemma 3 *Let $f \in C^2(0,1) \cap C(I)$. Then for any $\alpha \in (1, 2)$ $D^{\alpha} f$ exists and is continuous on I.*

The following results play an important role in the proof of our main result.

Theorem 4 *[2, Corollary 2.1 page 3] Let $f \in C^2(0,1)$ attains its minimum over the interval I at the point $t_0 \in (0,1)$ and $f'(0) \leq 0$. Then $D^{\alpha} f(t_0) \geq 0$ for any $\alpha \in (1, 2)$.*

We shall use the following notation. For $U, V \in C^2(I)$ $U \leq V$ means $U(t) \leq V(t)$ for all $t \in I$. Also, $[U, V] := \{v \in C^2(I); U \leq v \leq V\}$.

3. MAIN RESULT

In this section, we shall apply the lower and upper solutions method to develop an iterative technique to prove the existence of solutions to problem $(1)-(2)$.

Definition 3 *We call a function $\underline{u}$ a lower solution for problem $(1)-(2)$, if $\underline{u}\in C^2(I)$ and*

$$\begin{cases} D^{\alpha}\underline{u}(t)+\ f(t,\underline{u}(t))\geq 0, \quad t\in(0,1) \\ \underline{u}'(0)=0,\ \ \underline{u}(1)\leq\int\limits_0^1 g\left(\underline{u}(t)\right)dt. \end{cases}$$

Definition 4 *We call a function $\overline{u}$ an upper solution for problem $(1)-(2)$, if $\overline{u}\in C^2(I)$ and*

$$\begin{cases} D^{\alpha}\overline{u}(t)+\ f(t,\overline{u}(t))\leq 0, \quad t\in(0,1) \\ \overline{u}'(0)=0,\ \ \overline{u}(1)\geq\int\limits_0^1 g\left(\overline{u}(t)\right)dt. \end{cases}$$

Definition 5 *A solution of $(1)-(2)$ is a function $u\in C^2(I)$ that is both a lower solution and an upper solution of the problem.*

Define a truncation operator $\tau: C^2(I)\to[\underline{u},\overline{u}]$ by

$$\tau(y)=\max\{\underline{u},\min(y,\overline{u})\}.$$

Then $\tau(y)=\underline{u}$ if $y\leq\underline{u}$, $\tau(y)=y$ if $y\in[\underline{u},\overline{u}]$ and $\tau(y)=\overline{u}$ if $y\geq\overline{u}$. Moreover τ is a continuous and bounded operator. In fact, we have

$$\|\tau(u)\|_0\leq\max\left(\|\underline{u}\|_0,\|\overline{u}\|_0\right).$$

We now provide sufficient conditions on the nonlinearities f, g that will allow us to investigate problem $(1)-(2)$.
(**H1**) $f: I\times\mathbb{R}\to\mathbb{R}$ is continuous and satisfies $\left(f(t,v_1)-f(t,v_2)\right)(v_1-v_2)>0$, for all $t\in I$, $v_1>v_2$.
(**H2**) $g:\mathbb{R}\to\mathbb{R}$ is continuous and nondecreasing.

Theorem 5 *Assume that Problem $(1)-(2)$ has a lower solution $\underline{u}$, an upper solution $\overline{u}$ such that $\underline{u}(t)\leq\overline{u}(t)$, for all $t\in I$, and (H1), (H2) hold. Then Problem $(1)-(2)$ has at least one solution $u^*\in C[I]$ such that $\underline{u}(t)\leq u^*(t)\leq\overline{u}(t), t\in I$.*

Proof. The proof will be given in several steps.
Step1 : Modification of the problem. Let $\phi: I\times[\underline{u},\overline{u}]\to\mathbb{R}$ and $\psi:[\underline{u},\overline{u}]\to\mathbb{R}$ be defined, respectively, by

$$\phi(t,u)=f(t,\tau(u)),\ \ \psi(u)=g(\tau(u)).$$

It is clear that ϕ, ψ are continuous and bounded. Moreover ϕ satisfies (**H**1) and ψ satisfy (**H**2).

We consider the following modified boundary value problem

$$\begin{cases} D^{\alpha}u(t)+\phi(t,u(t))=0, \quad t\in(0,1), 1<\alpha\leq 2 \\ u'(0)=0,\ \ u(1)=\int\limits_0^1\psi(u(t))\,dt \end{cases}. \tag{5}$$

We will show that the modified problem (5) has at least one solution $u^*\in[\underline{u},\overline{u}]$. It follows that $\tau(u^*)=u^*$ so that $\phi(t,u^*)=f(t,u^*)$, $\psi(u^*)=g(u^*)$. This implies that u^* is a solution of our original problem $(1)-(2)$.

Step2. Let $b \in \mathbb{R}$. Consider the auxiliary problem

$$\begin{cases} D^{\alpha}u(t) + \phi(t,u(t)) = 0, \quad t \in (0,1), 1 < \alpha \leq 2 \\ u'(0) = 0, \;\; u(1) = b \end{cases} . \tag{6}$$

Claim. If (H1) is satisfied then (6) has a unique solution.
Step3. We develop an iterative method to show that the modified problem has at least one solution. Define a sequence $(u_k)_{k\in\mathbb{N}}$ in the following way. Let $u_0 = \underline{u}$ and for $k \geq 1$

$$\begin{cases} D^{\alpha}u_k(t) + \phi(t,u_k(t)) = 0, \quad t \in (0,1), 1 < \alpha \leq 2 \\ u_k'(0) = 0, \;\; u_k(1) = \int\limits_0^1 \psi(u_{k-1}(t))\, dt \end{cases} . \tag{7}$$

Notice that $u_k'(0)$ and $u_k(1)$ do not depend on the unknown function u_k. We see that problem (7) is similar to the previous auxiliary problem (6). Therefore, for each $k \in \mathbb{N}$, (7) has a unique solution $u_k \in \Omega$. It follows that (7) is equivalent to

$$u_k(t) = \int_0^1 G(t,s)\phi(s,u_k(s))ds + \int\limits_0^1 \psi(u_{k-1}(t))\, dt, \text{ for all } t \in I.$$

where $G(t,s)$ is Green's function corresponding to the linear homogeneous problem. This function exists because the homogeneous problem has only the trivial solution. It is given by

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} (1-s)^{\alpha-1}, & 0 \leq t < s \leq 1 \\ (1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \leq s < t \leq 1 \end{cases} .$$

Take limit as $k \to \infty$, using the continuity of ϕ and ψ, , we obtain

$$u^*(t) = \int_0^1 G(t,s)\phi(s,u^*(s))ds + \int\limits_0^1 \psi(u^*(t))\, dt, \text{ for all } t \in I.$$

Therefore

$$\begin{cases} D^{\alpha}u^*(t) + \phi(t,u^*(t)) = 0, \quad t \in (0,1), 1 < \alpha \leq 2 \\ u^{*'}(0) = 0, \;\; u^*(1) = \int\limits_0^1 \psi(u^*(t))\, dt \end{cases} . \tag{8}$$

Step 4. To complete the proof of our main result we need to prove that $\underline{u} \leq u^* \leq \overline{u}$, i.e. for all $t \in I$

$$\underline{u}(t) \leq u^*(t) \leq \overline{u}(t).$$

We deduce that for all $t \in I$

$$\phi(t,u^*(t)) = f(t,u^*(t)), \text{ and } \psi(u^*(t)) = g(u^*(t)).$$

Consequently,

$$\begin{cases} D^{\alpha}u^*(t) + f(t,u^*(t)) = 0, \quad t \in (0,1), 1 < \alpha \leq 2 \\ u'^*(0) = 0, \;\; u^*(1) = \int\limits_0^1 g(u^*(t))\, dt \end{cases} .$$

Finally, we see that u^* is the desired solution to our original problem. This completes the proof of our main result.

4. EXAMPLE

We consider the following boundary value problem

$$\begin{cases} D^{\frac{3}{2}}u(t)+e^{-u(t)}-1=0,\ t\in(0,1) \\ u'(0)=0,\ \ u(1)=\int\limits_0^1\left(1-e^{-u(t)}\right)dt. \end{cases} \tag{9}$$

We have $\alpha=3/2$, $f(t,u)=e^{-u}-1$, and $g(u)=1-e^{-u}$.

We see that $\underline{u}(t)=0$ is a lower solution for problem (9) and $\overline{u}(t)=1$ is an upper solution for problem (9). Applying Theorem 5, we see that the problem (9) has at least one solution $u^*\in C[I]$ with $0\le u^*(t)\le 1$, for all $t\in I$. Notice that we have obtained the existence of a nonegative solution.

5. REFERENCES

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