MARTINGALE METHODS FOR ANALYSING THE NON-MARKOVIAN MULTISERVER RETRIAL QUEUES

HOURIA OUKID

Department of Mathematics, University Saad Dahlab Blida1, Algeria

ABSTRACT

In this paper we use the martingale method for analysing a non-Markovian multiserver queue with n identical servers and losses. The orbite hase m waiting space and where customers that arrive when all servers are busy and the waiting space occupied are dropped and lost. Time intervals between possible retrials are assumed to have arbitrary distribution. Using the Doob-Meyer semimartingale decomposition, we provide an analysis of the general problem when the arrival and departure processes are quite general point processes and then solve it for particular special case when the arrival, departure and retrials processes are Markovian and the case when these processes are nonhomogeneous Poisson processes.

1. INTRODUCTION

Martingales are a quiet general class of stochastic processes for which the properties are based on those of the conditional mathematical expectation. The interpretation of this stochastic process is somewhat interesting in connection with Game Theory. Indeed a martingale's value can change ; however, its expectation remains constant in time. More important, the expectation of a martingale is unaffected by optional sampling. Martingales have been initially used in financial modeling, but it is surprising that such a powerful tool of probability theory has only been slowly taken up in queueing theory.

In the present paper we show how such an approach can be adapted to a non-Markovian multiserver retrial queueing system with losses. Such a queue can be used to model a switching center that allows a maximum of k simultaneous.

2. DESCRIPTION OF THE MODEL

Let us consider a multiserver queueing system with n servers having the following structure.

- Let $A(t)$ be the number of customer arrivals to the system in time interval $[0,t]$, $R(t)$ be the number of retrials and $D(t)$ be the number of customer departures in $[0,t]$.
- $A(t)$, $R(t)$ and $D(t)$ are point processes having strictly stationary and ergodic increments.
- There are n servers, and an arriving customer occupies one of free servers.
- The orbite has m waiting space.
- If upon arrival servers are busy, but the secondary queue, orbit, having at least one space free, then the customer occupies the orbit and retries more and more to occupy a server.
- A customer, who upon arrival finds all servers busy and the orbit occupied leave the system forever.
- Let $Q_1(t)$ denotes the number of customers in the queue at time t, which coincides with the number of busy servers at t, $Q_2(t)$ denotes the number of customers in orbite at time t and $Q_3(t)$ is the cumulated number of losses up to time t.

- Customers are served by one of the n idle servers ; the service times are mutually independent random variables, independent of the arrival process.
- All point processes considered in this paper are assumed to be right-continuous having left-side limits.
- The number of customers in the systems is always bounded

3. MARTINGALE REPRESENTATION

We can obtain a martingale representation for the queue-length process by using the Doob-Meyer semi-martingale decomposition and a technique due to Abramov [1]. We have the following basic representation :

$$Q_1(t)+Q_2(t)+Q_3(t)=A(t)-D(t),\ t\geq 0, \tag{1}$$

where the departure process $D(t)$ is defined with the help of the point processes $D_i(t),\ i=1,\dots,n$ as follows :

$$D(t)=\int_0^t \sum_{i=1}^{n} I\{Q_1(s^-)\geq i\}dD_i(s),\ t\geq 0, \tag{2}$$

where $I\{A\}$ is the indicator function of the event A.
Taking into account that the process $A(t)$ and $D_i(t)$, $i=1,2,\dots,n$ are semimartingales adapted to $\Im_n$. Denoting $\hat{A}(t)$ and $\hat{D}(t)$ the compensators of the processes $A(t)$ and $D(t)$ respectively. From Doob-Meyer decomposition theorem [e.g. Liptser and Shiryayev [5]

$$A(t)=\hat{A}(t)+M_A(t), \tag{3}$$

and

$$D(t)=\hat{D}(t)+M_D(t), \tag{4}$$

where $M_A(t)$ and $M_D(t)$ are local square integrable martingales. The compensators $\hat{A}(t)$ and $\hat{D}(t)$ have the integral representation

$$\hat{A}(t)=\int_0^t X(s)ds,\ t\geq 0, \tag{5}$$

and

$$\hat{D}(t)=\int_0^t Q_1(s)Y(s)ds,\ t\geq 0, \tag{6}$$

where $X=\{X(t):t\geq 0\}$ and $Y=\{Y(t):t\geq 0\}$ are adapted to the filtration $\Im$ and are called the stochastic intensity of the counting processes A and D respectively.
By virtue of (3) and (4), equation (1) can be rewritten in the form of Doob-Meyer semimartingale decomposition as follows :

$$Q_1(t)+Q_2(t)+Q_3(t)=\hat{A}(t)-\hat{D}(t)+M_A(t)-M_B(t). \tag{7}$$

Define the normalized processes $q_i(t)=\frac{1}{t}Q_i(t),\ \ i=1,2,3,$ $m_A(t)=\frac{1}{t}M_A(t)$ and $m_B(t)=\frac{1}{t}M_B(t)$.
Let us write the semimartingale decomposition for the queue-length process, from (7) we have :

$$q_1(t)+q_2(t)+q_3(t)=\frac{1}{t}\hat{A}(t)-\frac{1}{t}\hat{D}(t)+m_A(t)-m_B(t). \tag{8}$$

Therefore

$$q_1(t)+q_2(t)+q_3(t)=\frac{1}{t}\int_0^t X(s)ds-\frac{1}{t}\int_0^t Q_1(s)Y(s)ds+m_A(t)-m_B(t). \tag{9}$$

4. ANALYSIS OF THE LIMITING QUEUE-LENGTH DISTRIBUTIONS

In this section we derive equations for the following limits :

$$\lim_{t\to\infty}\frac{1}{t}E\int_0^t I\{Q_1(s^-)=i,Q_2(s^-)=j\}dA(s),\ i=1,2,...,n,\ j=0,1...m, \tag{10}$$

based on the Doob-Meyer semimartingale decomposition for the indicators of the queue-length process.
Let us denote

$$I_{i,j}(t)=I\{Q_1(t)=i,Q_2(t)=j\},\ i=0,1...n,\ j=0,1...m. \tag{11}$$

Taking into consideration that $I_{i,j}(t)=0$ if at least one of the indexes i or j is negative. Thus, we have the following theorem.

Theorem 1 *Given three independent counting processes $A(t)$, $D_i(t)$ and $R_j(t)$ having strictly stationary and ergodic increments, the limiting stationary frequencies of the queue-length processes $Q_1(t)$ and $Q_2(t)$ are given by :*

For i= 0,1,...,n-1, j = 0,1,...,m-1

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}E\int_0^t I_{i-1,j}(s^-)dA(s)+(i+1)\lim_{t\to\infty}\frac{1}{t}\int_0^t I_{i+1,j}(s^-)dD(s)\\&+(j+1)\lim_{t\to\infty}\frac{1}{t}E\int_0^t I_{i-1,j+1}(s^-)dR(s)\\&=\lim_{t\to\infty}\frac{1}{t}E\int_0^t I_{i,j}(s^-)dA(s)\\&+i\lim_{t\to\infty}\frac{1}{t}\int_0^t I_{i,j}(s^-)dD(s)+j\lim_{t\to\infty}\frac{1}{t}E\int_0^t I_{i,j}(s^-)dR(s)\end{aligned} \tag{12}$$

For i= 0,1,...,n-1, j=m

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}E\int_0^t I_{i-1,m}(s^-)dA(s)+(i+1)\lim_{t\to\infty}\frac{1}{t}\int_0^t I_{i+1,m}(s^-)dD(s)\\&=\lim_{t\to\infty}\frac{1}{t}E\int_0^t I_{i,m}(s^-)dA(s)+i\lim_{t\to\infty}\frac{1}{t}\int_0^t I_{i,m}(s^-)dD(s)+m\lim_{t\to\infty}\frac{1}{t}E\int_0^t I_{i,m}(s^-)dR(s)\end{aligned} \tag{13}$$

For i=n, j=0,1,...,m-1

$$\begin{aligned}&\lim_{t\to\infty}\frac{1}{t}E\int_0^t I_{n-1,j}(s^-)dA(s)+\lim_{t\to\infty}\frac{1}{t}E\int_0^t I_{n,j-1}(s^-)dA(s)\\&+(j+1)\lim_{t\to\infty}\frac{1}{t}E\int_0^t I_{n-1,j+1}(s^-)dR(s)\\&=\lim_{t\to\infty}\frac{1}{t}E\int_0^t I_{n,j}(s^-)dA(s)+n\lim_{t\to\infty}\frac{1}{t}\int_0^t I_{n,j}dD(s)\end{aligned} \tag{14}$$

For i=n, j=m

$$\lim_{t\to\infty}\frac{1}{t}E\int_0^t I_{n-1,m}(s^-)dA(s)+\lim_{t\to\infty}\frac{1}{t}E\int_0^t I_{n,m-1}(s^-)dA(s)$$

$$=n\lim_{t\to\infty}\frac{1}{t}\int_0^t I_{n,m}(s^-)dD(s) \tag{15}$$

5. SPECIAL CASES

In the following, we will give two special cases. Denote

$$P_{i,j}(t)=P\{Q_1(t)=i,\ Q_2(t)=j\},\ i=0,1,...,n,\ j=0,1,...,m-1, \tag{16}$$

$$P_{i,j}=lim_{t\to\infty}P\{Q_1(t)=i,\ Q_2(t)=j\},\ i=0,1,...,n\ j=0,1,...,m-1. \tag{17}$$

Corollary 2 *Assume that the processes $A(t)$, $D_i(t)$ and $R_j(t)$ are Poisson processes with rates λ, μ and θ respectively, then we have the following system of equations,*

For $i=0,\ j=0,1,...,m-1$

$$\lambda P_{0,j}=\mu P_{1,j}-j\theta P_{0,j}. \tag{18}$$

For $i=1,2,...,n-1,\ j=0,1,...,m-1$

$$(\lambda+i\mu+j\theta)P_{i,j}=\lambda P_{i-1,j}+(i+1)\mu P_{i+1,j}+(j+1)\theta P_{i-1,j+1}. \tag{19}$$

For $i=1,2,...,n-1,\ j=m$

$$(\lambda+i\mu+m\theta)P_{i,m}=\lambda P_{i-1,m}+(i+1)\mu P_{i+1,m}. \tag{20}$$

For $i=n,\ j=0,1,...,m-1$

$$(\lambda+n\mu)P_{n,j}=\lambda P_{n-1,j}+\lambda P_{n,j-1}+(j+1)\theta P_{n-1,j+1}. \tag{21}$$

For $i=n,\ j=m$

$$n\mu P_{n,m}=\lambda P_{n-1,m}+\lambda P_{n,m-1}. \tag{22}$$

Corollary 3 *Assume that the processes $A(t)$, $D_i(t)$, $R_j(t)$ are non homogeneous Poisson processes with rae $\lambda(t)$, $\mu(t)$ and $\theta(t)$ respectively, then we have the following system of equations, For $i=0,1,...,n-1,\ j=0,1,...,m,$*

$$lim_{t\to\infty}\frac{1}{t}\int_0^t[P_{i,j}(s)-P_{i-1,j}(s)]\lambda(s)ds=$$

$$-ilim_{t\to\infty}\frac{1}{t}\int_0^t P_{i,j}(s)\mu(s)ds-jlim_{t\to\infty}\frac{1}{t}\int_0^t P_{i,j}(s)\theta(s)ds$$

$$+(i+1)lim_{t\to\infty}\frac{1}{t}\int_0^t P_{i+1,j}(s)\mu(s)ds.$$

$$+(j+1)lim_{t\to\infty}\frac{1}{t}\int_0^t P_{i-1,j+1}(s)\theta(s)ds. \tag{23}$$

For $i=n,\ j=0,1,...,m-1,$

$$lim_{t\to\infty}\frac{1}{t}\int_0^t[P_{n,j}(s)-P_{n-1,j}(s)-P_{n,j-1}(s)]\lambda(s)ds=$$

$$-nlim_{t\to\infty}\frac{1}{t}\int_0^t P_{n,j}(s)\mu(s)ds-(j+1)lim_{t\to\infty}\frac{1}{t}\int_0^t P_{n-1,j+1}(s)\theta(s)ds \quad (24)$$

For $i=n,\ j=m,$

$$lim_{t\to\infty}\frac{1}{t}\int_0^t [P_{n-1,m}(s)-P_{n,m-1}(s)]\lambda(s)ds=$$
$$nlim_{t\to\infty}\frac{1}{t}\int_0^t P_{n,m}(s)\mu(s)ds. \quad (25)$$

6. CONCLUSION

In this paper, an analysis of non-Markovian multiserver retrial queueing system with losses having m space in orbite is provided with the aid of theory of martingales. The system of equations for this system is obtained, which will allow us to derive analytical results allowing us to provide a performance analysis.

References

1 Abramov, V. M. Analysis of multiserver retrial queueing system : A martingale approach and an algorithm of solution . *Annals of Opertions Research, 141, 19-50.* 2006.

2 Artalejo, J.R. *Accessible bibliography on retrial queues. Mathematical and computer Modelling, 30, 1-6.* . 1990.

3 Falin, G.I and Templeton, J.G.C. Retrial queues, *Chapman and Hall. Journal*, 1997.

4 Falin G.I. *A survey on Retrial queues, Queueing Systems, 7, 127-168*, 1990.

5 Liptser, R. S. And Shiryayev, A. N. *Theory of Martingales. Kluwer, Dordrecht.* 1989.