THE LIPSCHITZ WEAKLY $p$-NUCLEAR OPERATORS AND ITS INJECTIVE HULL

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ABSTRACT

Kim [10] introduced the notion of linear weakly $p$-nuclear operators. In this paper we introduce the notion of strongly Lipschitz weakly $p$-nuclear from a pointed metric space $X$ into a Banach space $E$. We show that they can be seen as a natural extension of the linear weakly $p$-nuclear operators, we transfer some properties of the linear case into the Lipschitz setting and we introduced this concept for $X$ and $E$ are pointed metric spaces (we say that "Lipschitz weakly $p$-nuclear operators"). In addition, we introduce the ideal of Lipschitz weakly quasi $p$-nuclear operators between pointed metric spaces and show that it coincides with the Lipschitz injective hull of the ideal of Lipschitz $p$-nuclear operators.

1. INTRODUCTION

Farmer and Johnson [7] have introduced the notion of Lipschitz $p$-summing operators there was an increasing interest in the study of different classes of Lipschitz functions between pointed metric spaces and Banach spaces. This is a true generalization of the concept of linear $p$-summing operators, since it is shown in [7] that the Lipschitz $p$-summing norm of a linear operator is the same as its $p$-summing norm. The paper [7] has motivated the study of Lipschitz version of different classes of bounded linear operators such as, Lipschitz $p$-integral and Lipschitz $p$-nuclear operators in [6]. Lipschitz$(p,r,s)$-integral operators and Lipschitz$(p,r,s)$-nuclear operators in [5]. Lipschitz compact operators and Lipschitz weakly compact operators in [9]. It turns out that almost all of the new classes of Lipschitz mappings studied fit in the framework of Lipschitz operator ideals.

In the present paper, we deal with a new class of Lipschitz operator ideals which extend the weakly $p$-nuclear operators of Kim and its injective hull [10].

The paper is organized as follows. After fixing some notation and establishing the basics of the theory of Lipschitz operators that we will use throughout the manuscript, in Sect. 3 we focus in the study of Lipschitz strongly weakly $p$-nuclear operators. We show that this type of operators fits in the theory of composition Banach Lipschitz operator ideal. This allows us to extend to the Lipschitz mappings setting the majority of the results obtained in the linear case. Also, from the factorization of linear weakly $p$-nuclear operators, we get a suitable factorization of the Lipschitz strongly weakly $p$-nuclear operators. As an application we extend some characterizations of the injective hull of the Lipschitz strongly weakly $p$-nuclear.

Finally, in the last section, we introduce the ideal of Lipschitz weakly $p$-nuclear operators and quasi weakly $p$-nuclear operators between pointed metric spaces and show that the Lipschitz injective hull of the ideal of Lipschitz weakly $p$-nuclear operators its coincides with the Lipschitz weakly quasi $p$-nuclear operators.
2. PRELIMINARIES

Our notation is standard. $X$ and $Y$ will be pointed metric spaces with a base point denoted by $0$ and metric will be denoted by $d$. We denote by $B_d$ the closure of the ball centered at $0$ with radius $1$. Also, $E$ and $F$ will stand for Banach spaces over the same field $\mathbb{K}$ (either $\mathbb{R}$ or $\mathbb{C}$) with dual spaces $E^*$ and $F^*$. A Banach space $E$ will be considered as pointed metric spaces with a base point $0$ and distance $d(x,x') = \|x-x'\|$. With $\text{Lip}_p(X,Y)$ we denote the set of all Lipschitz mappings from $X$ to $Y$ such that maps $0$ to $0$ and we put

$$\text{Lip}(T) = \inf \{ C > 0 : d(T(x),T(x')) \leq C d(x,x'); \forall x,x' \in X \}. $$

In particular, $\text{Lip}_0(X,E)$ is the Banach space of all Lipschitz mappings $T$ from $X$ to $E$ that vanish at $0$, under the Lipschitz norm $\text{Lip}(\cdot)$. When $E = \mathbb{K}$, $\text{Lip}_0(X,\mathbb{K})$ is denoted by $X^\theta$ and it is called the Lipschitz dual of $X$. The space of all linear operators from $E$ to $F$ is denoted by $\mathscr{L}(E,F)$ and it is a Banach space with the usual supremum norm. It is clear that $\mathscr{L}(E,F)$ is a subspace of $\text{Lip}_0(E,F)$ and, in particular, $E^*$ is a subspace of $E^\theta$.

Let $X$ be a metric space. A molecule on $X$ is a scalar valued function $m$ on $X$ with finite support that satisfies $\sum_{x \in X} m(x) = 0$. We denote by $\mathscr{M}(X)$ the linear space of all molecules on $X$. For $x,x' \in X$ the molecule $m_{xx'}$ is defined by $m_{xx'} = \chi_{\{x\}} - \chi_{\{x'\}}$, where $\chi_A$ is the characteristic function of the set $A$. For $m \in \mathscr{M}(X)$ we can write $m = \sum_{j=1}^n \lambda_j m_{x_jx_j'}$ for some suitable scalars $\lambda_j$, and we write

$$\|m\|_{\mathscr{M}(X)} = \inf \left\{ \sum_{j=1}^n |\lambda_j| d(x_j,x'_j) ; m = \sum_{j=1}^n \lambda_j m_{x_jx_j'} \right\},$$

where the infimum is taken over all representations of the molecule $m$. Denote by $\mathcal{A}(X)$ the completion of the normed space $(\mathscr{M}(X),\| \cdot \|_{\mathscr{M}(X)})$. This space was first introduced by Arens and Eells [4] in 1956. The terminology *Arens-Eells space* $\mathcal{A}(X)$ is due to Weaver [11]. The Arens-Eells space is also known as the Lipschitz-free Banach space of a metric space $X$. For more of this space, we refer the reader to the manuscript [9] and the reference therein. If we consider the canonical Lipschitz isometry $\delta_X : X \longrightarrow \mathcal{A}(X)$ given by $\delta_X(x) := m_{x_0x}$, $x \in X$, then for an operator $T \in \text{Lip}_0(X,E)$, there exists a unique linear map $T_L : \mathcal{A}(X) \longrightarrow E$ such that $T = T_L \delta_X$ and $\|T_L\| = \text{Lip}(T)$. The operator $T_L$ is referred to as the linearization of $T$ (see for instance [11] Theorem 2.2.4 (b))). The correspondence $T \longleftrightarrow T_L$ establishes an isometric isomorphism between the Banach spaces $\text{Lip}_0(X,E)$ and $\mathcal{A}(\mathcal{A}(X),E)$. In particular, the spaces $X^\theta$ and $\mathcal{A}(X)^\theta$ are isometrically isomorphic via the linearization $R(f) := f_L$, where $f_L(m) = \sum_{x \in X} f(x)m(x)$.

Following [11], an ideal of Lipschitz mappings $\mathscr{J}_{\text{Lip}}$ is a subclass of the class of all Lipschitz mappings between pointed metric spaces and Banach spaces such that for a metric space $X$ and Banach space $E$, the components

$$\mathscr{J}_{\text{Lip}}(X,E) := \text{Lip}_0(X,E) \cap \mathscr{J}_{\text{Lip}},$$

is a vector subspace of $\text{Lip}_0(X,E)$ that is invariant by the composition of a linear operator on the right and Lipschitz operator on the left and which contains the Lipschitz finite rank mappings type.

Let $1 \leq p \leq \infty$, we write $p^*$ the conjugate index of $p$, that is $1/p + 1/p^* = 1$. As usual, when $p = 1, p^* = \infty$. Let $(f_j)_j$ be a sequence in $X^\theta$.

- Following [5], the sequence $(f_j)_j$ is Lipschitz $\omega^*-p$-summable if there is a constant $C$ such that for all $n \in \mathbb{N}$ and for all $x,x' \in X$ we have

$$\left\| (f_j(x) - f_j(x'))^n_j \right\|_p \leq C d(x,x').$$

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The smallest such constant $C$ will be denoted by $\| (f_j)_j \|_{L^p,\omega^*}^p$ and $\ell_p^{L^p,\omega^*}(\mathcal{X})$ will denote the set of Lipschitz $\omega^*$-p-summable sequences in $\mathcal{X}$. Clearly

$$\| (f_j)_j \|_{L^p,\omega^*}^p = \sup_{x,x' \in \mathcal{X}} \frac{\| (f_j(x) - f_j(x'))_j \|_p}{d(x,x')}.$$ 

- A sequence $(f_j)_j$ in $\mathcal{X}$ is called Lipschitz $\omega^*$-unconditionally p-summable if $(f_j)_j$ is in $\ell_p^{L^p,\omega^*}(\mathcal{X})$ and

$$\sup_{x,x' \in \mathcal{X}} \frac{\sum_{j=1}^{+\infty} |f_j(x) - f_j(x')|^p}{d(x,x')} \to 0.$$ 

The set of all Lipschitz $\omega^*$-unconditionally p-summable sequences in $\mathcal{X}$ is denoted by $\ell_p^{L^p,\omega^*}(\mathcal{X})$.

It is a well known result that the canonical correspondence $T \mapsto (\langle \varepsilon^* x, T(y) \rangle)_j$ provides an isometric isomorphism of $\text{Lip}_0(X,\ell_p)$ onto $\ell_p^{L^p,\omega^*}(\mathcal{X})$ [3 Lemma 2.4]. Also, $\ell_p^{L^p,\omega^*}(\mathcal{X}), \| \cdot \|_{L^p,\omega^*} = \ell_p^{\omega^*}(\mathcal{X}), \| \cdot \|_p$.

Kim [10] introduced an ideal of weakly $p$-nuclear operators. Let $1 \leq p \leq \infty$ we say that an operator $T : E \to F$ is weakly $p$-nuclear if it is represented as

$$T = \sum_{n=1}^{+\infty} x_n^* \otimes y_n,$$

where $(x_n^*)_n \in \ell_p^0(E^*) (c_0^0(E^*))$ when $p = \infty$ and $(y_n)_n \in \ell_p^0(F) (c_0^0(F))$ when $p = 1$. We denote the space of all weakly $p$-nuclear operator from $E$ to $F$ by $\mathcal{N}_{ap}(E,F)$ and define a norm on $\mathcal{N}_{ap}(E,F)$ by

$$\| T \|_{\mathcal{N}_{ap}} := \inf \| (x_n^*_n) \|_p \| (y_n)_n \|_p,$$

where the infimum is taken over all such weakly $p$-nuclear representations of $T$.

Let $1 \leq p \leq \infty$ and let $T : E \to F$ be a linear map.

1. $T \in \mathcal{N}_{ap}(E,F)$ if and only if there exist $R \in \mathcal{L}(E,\ell_p)$ and $S \in \mathcal{L}(\ell_p,F)(\ell_p)$ is replaced by $c_0$ if $p = \infty$ such that $T = SR$.

   In this case,

   $$\| T \|_{\mathcal{N}_{ap}} = \inf \| R \| \| S \|,$$

   where the infimum is taken over all such factorizations.

2. $T \in \mathcal{N}_{ap}(E,F)$ if and only if there exist $R \in \mathcal{N}_{ap}(E,\ell_p)$ and $S \in \mathcal{N}_{ap}(\ell_p,F)(\ell_p)$ is replaced by $c_0$ if $p = \infty$ such that $T = SR$

3. **MAIN RESULTS**

3.1. **The Lipschitz weakly $p$-nuclear operators between metric space and Banach space.**

**Definition 1** Let $X$ be metric space and $E$ be a Banach space. For $\frac{1}{p} + \frac{1}{p'} = 1$ and $T \in \text{Lip}_0(X,\mathcal{E})$, we say that $T : X \to E$ is strongly Lipschitz weakly $p$-nuclear if $T$ can be written in the form:

$$T x = \sum_{j=1}^{+\infty} f_j(x) y_j , \forall x \in X$$

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where \((f_n)_n \in \ell^0_p(X^0)(c_0(X^0))\) when \(p = \infty\) and \((y_n)_n \in \ell^0_p(E)(c_0(E))\) when \(p = 1\). Also, the set of all strongly Lipschitz weakly p- nuclear will be denoted by \(\mathcal{A}_{\omega p}^L(X, E)\) and we set

\[
\|T\|_{\mathcal{A}_{\omega p}^L} = \inf \left\{ \|f_n\|_{\ell^0_p} \|y_n\|_p : (f_n)_n \in \ell^0_p(X^0)(c_0(X^0)) \text{ when } p = \infty \right\}
\]

with the infimum taken over all representations of \(T\) as in [1].

**Proposition 1** Let \(1 \leq p \leq q \leq \infty\), \(X\) be a pointed metric and \(E\) be a Banach space. Then, the strongly Lipschitz weakly p- nuclear are strongly weakly q - nuclear and \(\|T\|_{\mathcal{A}_{\omega p}^L} \leq \|T\|_{\mathcal{A}_{\omega q}^L}\).

**Theorem 2** Let \(1 \leq p \leq \infty\) and \(T \in \mathcal{A}_{\omega p}^L(X, E)\). Then \(T \in \mathcal{A}_{\omega p}^L(X, E)\) if and only if there exist \(R \in \mathcal{A}(X, \ell_p)(R \in \mathcal{A}(X, c_0)\) when \(p = \infty\)) and \(S \in \mathcal{L}(\ell_p, Y)\) such that the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{T} & E \\
\downarrow R & & \downarrow S \\
\ell_p & & \\
\end{array}
\]

In this case:

\[
\|T\|_{\mathcal{A}_{\omega p}^L} := \inf \|R\| \|S\|.
\]

**Theorem 3** Let \(X\) be a pointed metric space, \(E\) be a Banach space and \(1 \leq p \leq \infty\). An operator \(T \in \mathcal{A}_{\omega p}^L(X, E)\) if and only if its Linearization \(T_L : \mathcal{A}(X) \rightarrow E\) is linear weakly p - nuclear. Moreover, we have

\[
\|T\|_{\mathcal{A}_{\omega p}^L} = \|T\|_{\mathcal{A}_{\omega p}}
\]

By Theorem [3] and the criterion in [1], we have the following.

**Proposition 4** The class \(\mathcal{A}_{\omega p}^L\) is the Banach Lipschitz operator ideal generated by the composition method from the Banach operator ideal \(\mathcal{N}_{\omega p}\). In other words

\[
\mathcal{A}_{\omega p}^L(X, E) = \mathcal{N}_{\omega p} \circ \mathcal{Lip}_0(X, E)
\]

for every pointed metric space \(X\) and every Banach space \(E\).

As \(\mathcal{A}_{\omega p}^L\) is a Banach Lipschitz operator ideal of composition type (Proposition 4), then from Proposition 2.4 in [2] and Theorem 4.4 in [10] we get

**Corollary 5** Let \(X\) be pointed metric space and \(E\) be a Banach space. Then

\[
(\mathcal{A}_{\omega p}^L)^{\text{inj}} = (\mathcal{N}_{\omega p} \circ \mathcal{Lip}_0)^{\text{inj}} = (\mathcal{N}_{\omega p})^{\text{inj}} \circ \mathcal{Lip}_0 = \mathcal{A}_{\omega p}^Q \circ \mathcal{Lip}_0.
\]

**Proposition 6** Let \(X\) be pointed metric space, \(E\) be a Banach space and \(1 \leq p \leq \infty\), the following are equivalent.

\(1\) \(T \in (\mathcal{A}_{\omega p}^L)^{\text{inj}}\).
(2) There exists a sequence \((f_n)_n \in \ell_p^{\omega^*}(X^\#)\) such that
\[
\| \sum_{j=1}^m \lambda_j (T x_j - T x_j') \| \leq \left( \sum_{n=1}^\infty \left( \sum_{j=1}^m \lambda_j (f_n(x_j) - f_n(x_j')) \right)^p \right)^{\frac{1}{p}}
\] (2)
for all \(x_j, x_j' \in X, 1 \leq j \leq m\). In such case, we put
\[
\| T \|_{(\mathcal{N}(\mathcal{A}_{\omega^*}^L), \omega^*)_{\infty}} = \inf \left\{ \| (f_n)_n \|_{\ell_p^{\omega^*}} : (f_n)_n \text{satisfying (2)} \right\}
\]

3.2. The Lipschitz weakly \(p\)-nuclear operators between two metric spaces

Definition 2 Let \(X\) and \(Y\) be pointed metric spaces and \(1 \leq p \leq \infty\). A mapping \(T \in \text{Lip}_0(X, Y)\) is called Lipschitz weakly \(p\)-nuclear if there exist \(R \in \text{Lip}_0(X, \ell_p)\) and \(S \in \text{Lip}_0(\ell_p, Y)\) (\(\ell_p\) is replaced by \(c_0\) if \(p = \infty\)) such that the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{T} & Y \\
\downarrow R & & \downarrow S \\
\ell_p & & \\
\end{array}
\]

In this case:
\[
\mathcal{N}^L_{\omega^*}(T) := \text{Lip}(S) \cdot \text{Lip}(R)
\]

A proof similar to [6, Theorem 2.1] show that the Lipschitz weakly \(p\)-nuclear norm is equal to the weakly \(p\)-nuclear from a linear operator from a separable space to a dual space.

Proposition 7 Let \(1 \leq p < \infty\), and Let \(T\) be a bounded linear operator from a separable Banach space \(X\) into a dual space \(Y\). Then
\[
\| T \|_{\mathcal{N}^L_{\omega^*}} = \| T \|_{\mathcal{N}^L_{\omega^*}}
\]

Definition 3 Let \(1 \leq p < \infty\) and let \(X, Y\) be a pointed metric spaces. A mapping \(T \in \text{Lip}_0(X, Y)\) is called Lipschitz weakly quasi \(p\)-nuclear if there exists a sequence \((f_n)_n \in \ell_p^{\omega^*}(X^\#)\) such that
\[
d(T(x), T(x')) \leq \left( \sum_{n=1}^\infty |f_n(x) - f_n(x')|^p \right)^{\frac{1}{p}}
\] (3)
for all \(x, x' \in X\). We denote by \(\mathcal{D}^{L}_{\omega^*}(X, Y)\) the space of all weakly quasi \(p\)-nuclear Lipschitz mappings between pointed metric spaces \(X\) and \(Y\).

In such case, we put
\[
\| T \|_{\mathcal{D}^{L}_{\omega^*}} = \inf \left\{ \| (f_n)_n \|_p : (f_n)_n \text{satisfying (3)} \right\}
\]

Proposition 8 For \(1 \leq p < \infty\), \(\mathcal{D}^{L}_{\omega^*}\) is a Lipschitz operator ideal.

Lemma 9 Let \(1 \leq p < \infty\). If \(T \in \mathcal{D}^{L}_{\omega^*}(X, \ell_\infty(\Gamma))\) then \(T \in \mathcal{A}^{L}_{\omega^*}(X, \ell_\infty)\) and \(\| T \|_{\mathcal{A}^{L}_{\omega^*}} \leq \| T \|_{\mathcal{D}^{L}_{\omega^*}}\)

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In [3], the authors introduced the injective hull of a Lipschitz operator ideal.

Let $\mathcal{F}_{\text{Lip}}$ be a Lipschitz operator ideal between pointed metric spaces. For pointed metric spaces $X$ and $Y$, a Lipschitz operator $T \in \text{Lip}_0(X,Y)$ belongs to the Lipschitz injective hull of $\mathcal{F}_{\text{Lip}}$ if there exists a pointed metric space $Z$ and a Lipschitz operator $S \in \text{Lip}(X,Z)$ such that $d(Tx,Tx') \leq d(Sx, Sx')$ for all $x, x' \in X$. The class of all operators from $X$ to $Y$ which belongs to the Lipschitz injective hull of $\mathcal{F}_{\text{Lip}}$ will be denoted by $\mathcal{F}^{\text{Linj}}_{\text{Lip}}(X,Y)$.

**Theorem 10** Let $1 \leq p < \infty$ and let $X$ and $Y$ be pointed metric spaces. A Lipschitz operator $T \in \mathcal{L}^p_{\text{ap}}(X,Y)$ if and only if $T \in \mathcal{L}^p_{\text{ap}}(X,Y)$.

In this case,

$$\|T\|_{\mathcal{L}^p_{\text{ap}}(X,Y)} = \|T\|_{\mathcal{L}^p_{\text{ap}}(X,Y)}$$

**Proposition 11** Let $1 \leq p < \infty$, $E$ and $F$ be Banach spaces and $T \in \mathcal{L}(E,F)$. Then

$$\mathcal{N}^\text{Linj}_{\mathcal{L}^p_{\text{ap}}}(E,F) = \mathcal{N}^\text{Linj}_{\mathcal{L}^p_{\text{ap}}}(E,F) \subseteq \mathcal{N}^\text{Linj}_{\mathcal{L}^p_{\text{ap}}}(E,F).$$

The equality holds when $E$ is separable. Moreover, the norms $\|\cdot\|_{\mathcal{L}^p_{\text{ap}}(E,F)}$ and $\|\cdot\|_{\mathcal{L}^p_{\text{ap}}(E,F)}$ coincides.

### 4. CONCLUSIONS

In this work, we introduce the class of Lipschitz weakly $p$-nuclear operators, and we prove a several properties related to Lipschitz weakly $p$-nuclear operators. Also we show that this class is a true generalization of the linear case.

### 5. REFERENCES


[6] D.Chen and B.Zheng, Lipschitz $(p,r,s)$-integral operators and Lipschitz $p$-nuclear operators, Nonlinear Analysis 75(13)(2012),5270-5282


