TVD WAF SCHEME WITH PVRS RIEMANN SOLVER FOR THE DRIFT-FLUX EQUATIONS OF TWO-PHASE FLOWS UNDER ISOTHERMAL CONDITIONS

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ABSTRACT

This paper concentrated on the extension of the TVD version of Weighted Average Flux (WAF) scheme for the numerical simulation of two-phase gas-liquid flow for drift-flux model under isotermal condition. The model becomes a hyperbolic system of conservation laws with realistic closure relations where both phases are strongly coupled during their motion. Based on a PVRS Riemann solver for the drift-flux model, the model is numerically solved by the WAF scheme. The accuracy of the proposed numerical algorithm is verified by comparing it with other numerical methods compared to the exact solutions through a set of different test cases mentioned in the literature [3]. Simulation results appear appropriate agreement of tvd WAF scheme together with the exact solutions. we note that the behaviour of WAF scheme is encouraging, more accurate and fast than other numerical methods.
Key words : WAF scheme ; approximation ; drift-flux model ; Riemann problem ; numerical simulation

1. INTRODUCTION

The drift model that governs the equations presented in this paper is simples of the two-fluid model type. Moreover, the model has simpler expression form with low computational cost for predicting basic flow properties. the drift-flux model is frequently considered for the numerical approximation of practical cases related to two-phase flows. As regard as to the numerical approximations of such models, the model requires advanced and specialised methods due to their mathematical and physical. The model equations are solved exactely by [S. Kuila, T. R. Sekhar and D. Zeidan] in the literature [2, 3], such the analytical solution suggested here is based on the Riemann problem and is constructed under the assumption of isothermal conditions. In [2, 3], Godunov methods of centred-type are extended to both isothermal and isentropic gas-liquid two-phase flow employing the drift-flux model and validated with the developed exact Riemann solvers. The objective of this paper is to extend another high-order Godunov method of single-phase flow to the isotermal drift-flux model. In this context, the Weighted Average Flux (WAF) approximation proposed initially in [1] is extended for simulating compressible gas and liquid two-phase flows. It is worth to note that the WAF scheme is applied to approximate shock wave propagations appearing in two-phase flows governed by the two-fluid model in [4, 5, 6, 8]. This scheme is completely different from the numerical methods which have been used before to resolve the drift flow model, and almost similar to what was used in [9]. The WAF scheme for the isentropic drift-flux model of compressible two-phase flows. The WAF scheme is a generalisation of the widely known Godunov first-order upwind scheme for the non-linear conservation laws. On this basis, we propose to approximate this model with the TVD version of WAF scheme based on the PVRS approximation Riemann solver originally used in literature [1]. It is a clear scheme that has second-order accuracy in space and time. Moreover, the feature of the scheme is that it is based on the fact that all the distinct fields are included. In the following sections, the used WAF scheme is presented in detail with version TVD and make a description of the user model. In the last section, the results obtained are presented.

2. NUMERICAL RESOLUTION APPROACH

The drift-flux model can be written in the following form without taking into account the presence source terms [2, 3] :

$$\frac{\partial \mathbb{U}}{\partial t}+\frac{\partial \mathbb{F}(\mathbb{U})}{\partial x}=0, \quad x\in\mathbb{R} \quad \text{and} \quad t\in\mathbb{R}^+, \tag{1}$$

in which the state vector of conservative variables $\mathbb{U}$ and the fluxes vector $\mathbb{F}$ are :

$$\mathbb{U}=\begin{pmatrix}\alpha_g\rho_g\\ \alpha_l\rho_l\\ (\alpha_g\rho_g+\alpha_l\rho_l)u\end{pmatrix} \quad \text{and} \quad \mathbb{F}(\mathbb{U})=\begin{pmatrix}\alpha_g\rho_g u\\ \alpha_l\rho_l u\\ (\alpha_g\rho_g+\rho_l\alpha_l)u+P\end{pmatrix},$$

which constitute the gas and liquid mass balance equations, as each component is isothermal, the mixture pressure [Kuila et al. (2015)] given by

$$P=a_g^2P_g+a_l^2P_l, \tag{2}$$

The system (1) in a nonconservative form writen as

$$\frac{\partial \mathbb{V}}{\partial t}+\mathbb{A}\frac{\partial \mathbb{V}}{\partial x}=0, \tag{3}$$

with

$$\mathbb{V}=\begin{pmatrix}\alpha_g\rho_g\\ \alpha_l\rho_l\\ u\end{pmatrix} \quad \textit{and} \quad \mathbb{A}(\mathbb{V})=\begin{pmatrix}u & 0 & \alpha_g\rho_g\\ 0 & u & \alpha_l\rho_l\\ \dfrac{a_g^2}{\alpha_g\rho_g+\alpha_l\rho_l} & \dfrac{a_l^2}{\alpha_g\rho_g+\alpha_l\rho_l} & u\end{pmatrix} \tag{4}$$

It has been shown in the literature [3] that system must be strictly deterministic with a complete eigenstructure for an isothermal state. Where eigenvalues are given by :

$$\lambda_1=u-\sqrt{\frac{a_g^2P_g+a_l^2P_l}{\alpha_g\rho_g+\alpha_l\rho_l}}, \quad \lambda_2=u \quad \text{and} \quad \lambda_3=u+\sqrt{\frac{a_g^2P_g+a_l^2P_l}{\alpha_g\rho_g+\alpha_l\rho_l}}. \tag{5}$$

the numerical precision of system (1) can be approximated by the conservative and explicit finite volume , and taking into consideration the initial and boundary conditions within a control volume $[x_{i-\frac{1}{2}},x_{i+\frac{1}{2}}]\times[t^n,t^{n+1}]$ as :

$$\mathbb{U}_i^{n+1}=\mathbb{U}_i^n+\frac{\Delta t}{\Delta x}\left[\mathbb{F}_{i-\frac{1}{2}}-\mathbb{F}_{i+\frac{1}{2}}\right], \tag{6}$$

in which $\mathbb{U}_i^n$ is the space average approximation at time n in the ith computational cell and $\mathbb{F}^n_{i+\frac{1}{2}}$ is the numerical inter cells flux at time n in the ith computational cell. The distance between two cross sections is Δx and Δt is the time step calculated using :

$$\Delta t = \mathscr{C}\,\frac{\Delta x}{\lambda_{\max}^{(n)}}, \quad 0 \le \mathscr{C} \le 1, \tag{7}$$

where $\mathscr{C}$ is the Courant-Friedrichs-Lewy (CFL) number. $\lambda_{\max}^{(n)}$ given by

$$\lambda_{\max}^{(n)} = \max_i \left\{|\lambda_i|\right\}, \tag{8}$$

is the maximum wave speed at the current time level and λ_i are eigenvalues corresponding to sound waves.

For the method using , The numerical fluxes at the cell face are computed by solving the local Riemann problem for system (1) with the initial data :

$$\mathbb{U}\big(x,t=0\big) = \begin{cases} \mathbb{U}_i^n & \text{if} \qquad x \le x_0, \\ \mathbb{U}_{i+1}^n & \text{if} \qquad x > x_0, \end{cases} \tag{9}$$

in which x_0 is the initial data discontinuity. The WAF scheme [1] is a second-order approach of accuracy in time and space without data reconstruction, which is developed in the framework of conservative systems. In this scheme, the intercell numerical fluxes are calculated through the following integral average form of $\mathbb{F}(\mathbb{U})$ at the half-time step [1] :

$$\mathbb{F}^{\mathtt{WAF}}_{i+\frac{1}{2}} = \frac{1}{\Delta x}\int_{\frac{-\Delta x}{2}}^{\frac{\Delta x}{2}} \mathbb{F}\left[\mathbb{U}_{i+\frac{1}{2}}\left(x,\frac{\Delta t}{2}\right)\right] dx, \tag{10}$$

in which $\mathbb{U}_{i+\frac{1}{2}}$ is the solution of the Riemann problem with the initial states (9). This solution is based on the information about the complete wave structure of system (1) making use of the wave jumps. The solution then is based on primitive variable PVRS approximate Riemann solvers for the drift-flux model. For solving linear hyperbolic systems with constant coefficients and and apply the pvrs rieman solver was originally mentioned in the literary [1], we assume that $\mathbb{A}(\mathbb{V})$ to be constant and the initial data $\mathbb{V}_L$, $\mathbb{V}_R$ and the solution are close to a constant state $\bar{\mathbb{V}}$, so

$$\bar{\mathbb{A}} = \mathbb{A}(\bar{\mathbb{V}}) \tag{11}$$

Then the approximation of the Riemann problem for system treat as the system in conservative form by

$$\frac{\partial \mathbb{V}}{\partial t} + \frac{\partial \mathbb{F}(\mathbb{V})}{\partial x} = 0, \tag{12}$$

$$\mathbb{F}(\mathbb{V}) \equiv \bar{\mathbb{A}}\mathbb{V}$$

and within $\bar{\lambda}_i$

$$\Delta \mathbb{F} = \bar{\lambda}_i \Delta \mathbb{V} \tag{13}$$

Direct application of waves 1 and 3 and after some algebraic manipulation we get the complete solution. The WAF integral average (10) can be written in the form of wave structure as :

$$\mathbb{F}^{\mathtt{WAF}}_{i+\frac{1}{2}} = \frac{1}{2}\Big(\mathbb{F}_i + \mathbb{F}_{i+1}\Big) - \frac{1}{2}\sum_{k=1}^{N} \operatorname{sign}(\mathscr{C}_k)\,\phi^{(k)}_{i+\frac{1}{2}}\,\Delta\mathbb{F}^{(k)}_{i+\frac{1}{2}}. \tag{14}$$

where

$$\phi^{(k)}_{i+\frac{1}{2}} = \phi_{i+\frac{1}{2}}(\phi(r^k, |\mathscr{C}|) \quad and \quad \mathscr{C}_k = \lambda_k \frac{\Delta t}{\Delta x}, \tag{15}$$

$\mathscr{C}_k$ is the wave CFL number for wave k of speed λ_k, $\phi^{(k)}_{i+\frac{1}{2}}$ is the WAF limiter employed to apply the Total Variation Diminishing (TVD) condition to the flux during the computation and to suppress any spurious oscillations near discontinuities and high gradients, Although there are many choices for limiters, it is found that the SUPERBEE limiter of Roe [7] is the best suited to the current investigation. The WAF limiter, where the SUPERBE function is applied in this work, is given as follow :

$$\phi(r^k, |\mathscr{C}|) = \begin{cases} 1 & \text{if} \quad r^k \leq 0, \\ 1 - 2(1 - |\mathscr{C}_k|)\, r^k & \text{if} \quad 0 \leq r^k \leq \frac{1}{2}, \\ |\mathscr{C}_k| & \text{if} \quad \frac{1}{2} \leq r^k \leq 1 \\ 1 - (1 - |\mathscr{C}_k|)\, r^k & \text{if} \quad 1 \leq r^k \leq 2, \\ 2|\mathscr{C}_k| - 1 & \text{if} \quad r^k \geq 2, \end{cases} \tag{16}$$

in which

$$r^{(k)} = \begin{cases} \dfrac{\Delta\chi^k_{i-\frac{1}{2}}}{\Delta\chi^k_{i+\frac{1}{2}}} & \mathscr{C}_k > 0, \\ \dfrac{\Delta\chi^k_{i+\frac{3}{2}}}{\Delta\chi^k_{i+\frac{1}{2}}} & \mathscr{C}_k < 0 \end{cases} \tag{17}$$

where the choice of χ is liquid or gas density as follow :

$$\chi_g = \alpha_g \rho_g \quad \text{or} \quad \chi_l = \alpha_l \rho_l. \tag{18}$$

3. COMPUTATIONAL RESULTS

The objective of this paper is to extend a weighted average flow (WAF)scheme of the drift flow model of two phase with using PVRS approximation of Riemann solver. To demonstrate the numerical accuracy and computational efficiency of the current WAF scheme, some published test cases have been carried out are performed. [2], where the initial data of this cases are shown in table 1.

TABLE 1 – Initial data for Test 1 as in [3].

Test		ρ_1	ρ_2	u	a_g	a_l
Test 1	Left data	50.0	1000.0	-100.00	500.0	100.0
	Right data	50.0	1000.0	100.00	500.0	100.0
Test 2	Left data	6.0	1.0	0.0	10.0	7.0
	Right data	1.0	1.0	0.0	10.0	7.0
Test 3	Left data	0.1	0.1	200.0	700.0	300
	Right data	0.1	0.1	-200.0	700.0	300
Test 4	Left data	500	500	0.1	50.0	20.0
	Right data	600	600	1.0	50.0	20.0

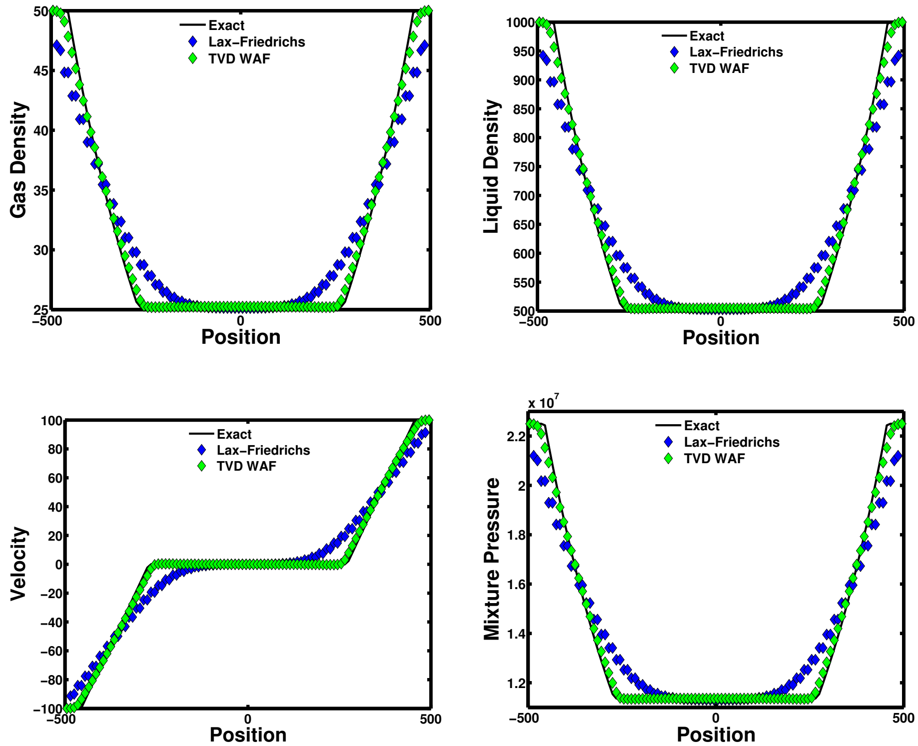

FIGURE 1 – Test 1. Comparison of TVD WAF scheme with the Lax-Friedrichs methods together with the exact Riemann solver at time t= 1.85.

These include : (i) Test 1 : pure rarefaction and the solution is presented at time t = 1.85 are shown in Figure (1) ; (ii) Test 2 : shock waves : The solution is composed of a left shock wave, a contact discontinuity and right shock wave ; where it's displayed at time t= 0.015 are illustrated in Figure (2) ; and (iii) Test 3 : left shock wave, a contact discontinuity and right rarefaction wave where the solution it's displayed at time t= 0.17 are illustrated in Figure (3). In all test cases, we compare results provided by numerical resolution with previously pvrs approximation Riemann solver for drift flow model validation.

In following test cases, the drift-flux model is solved in a computational domain of $[-10,10]$ with take the initial data discontinuity $x_0 = 0$ and CFL parameter 0.9 is always fixed together with transmissive boundary conditions along with the SUPERBEE limiter are employed through out the simulations. The different symbols in all test cases represent three different methods, namely, the WAF, Lax-Friedrichs and TVD slic methods where the solid line refers to the exact Riemann solver of [2].

4. REFERENCES

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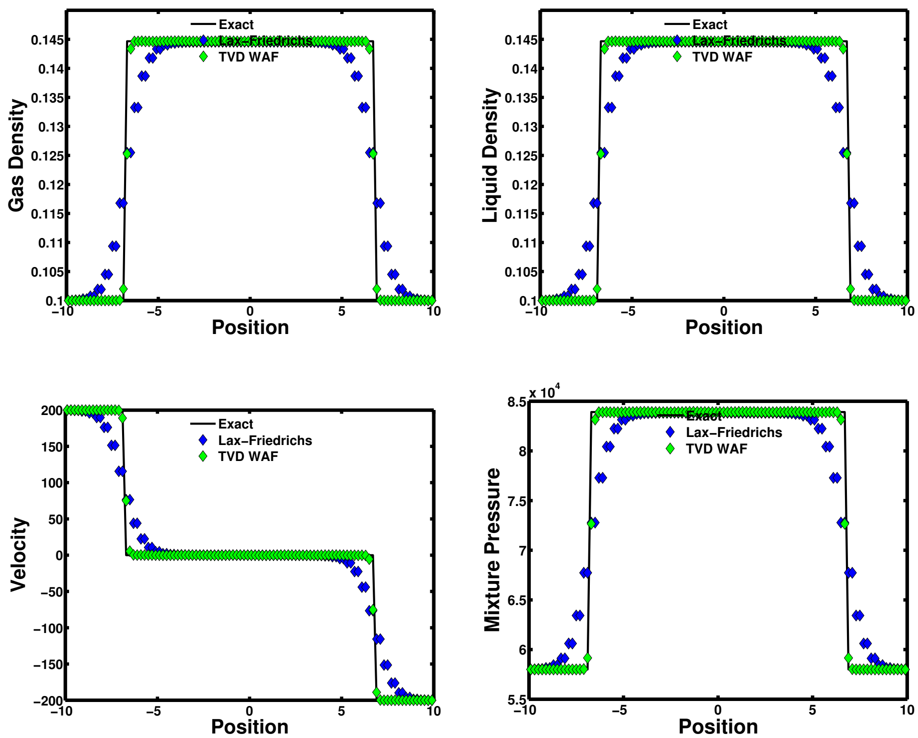

FIGURE 2 – Test 2. Two different numerical methods, namely, TVD WAF and Lax-Friedrichs methods are compared with the exact Riemann solver at time t= 0.015 for the velocity and mixture pressure. Where a CFL = 0.9 and 100 cells are employed for the numerical resolutions

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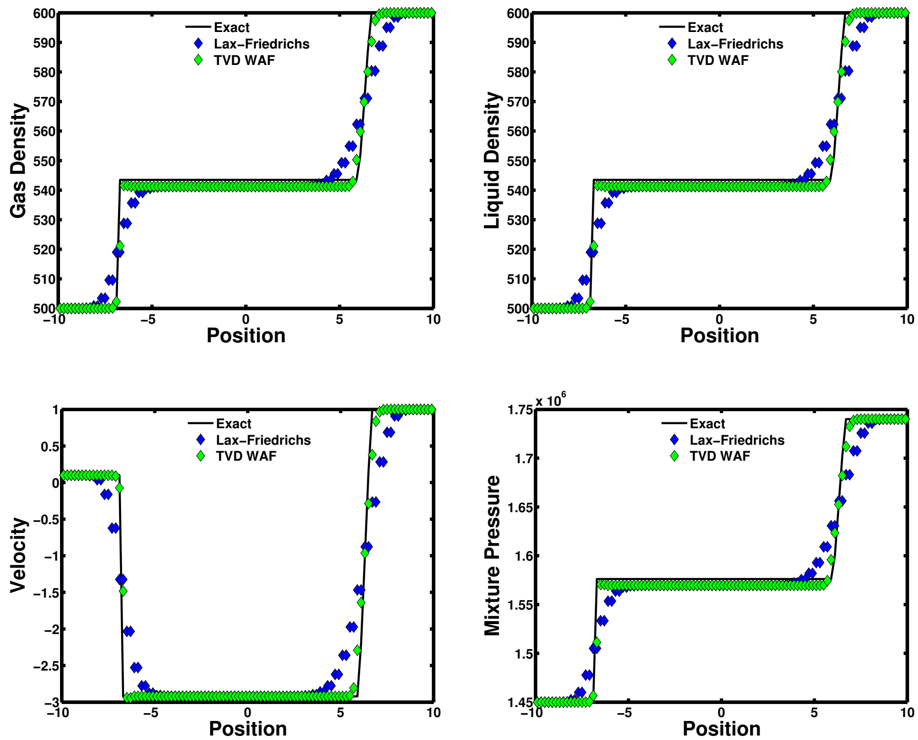

FIGURE 3 – Test 3.Comparison of numerical methods, namely, TVD WAF and Lax–Friedrichs methods with the exact Riemann solver at time t = 0.17, CFL =0.9 and with 100 cells.

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