EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR SYSTEM OF TIME-INVARIANT FRACTIONAL DIFFERENTIAL EQUATIONS

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ABSTRACT

In this paper, using the Schauder fixed point theorem, the existence and uniqueness of the solution of the Cauchy type system of non linear fractional differential equations is established, in which the fractional derivative in the sense of Caputo is used. Also, we propose the periodicity of the long-term solution of the fractional dynamical system. Finally, an example has been illustrated to support the proof and make it tangible.

Keywords : fractional differential equations, existence, uniqueness, Caputo's derivative, periodic solution. IEEEkeyword

1. INTRODUCTION

In recent years, fractional calculus has attracted the attention of researchers in engineering, biology, economics and many other fields. In this article, we have studied the initial value problem of fractional differential equations (IVP)

$$\begin{cases} ({}^{C}D^{\alpha}u)(x) = f(x,u(x)); \\ u(x_0) = u_0, \end{cases} \tag{1}$$

where the fractional derivative is in the sense of Caputo's definition, the function $f(x,u) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is called vector field, and the dimension $n \geqslant 1$. Especially , $\mathbb{R}^n$ gives the appropriate a proper norm $\|.\|$ becomes a complete metric space.
Let $J = [x_0 - a, x_0 + a], B = \{u \in \mathbb{R}^n | \ \|u - u_0\| \leqslant b\}$ and $E = \{(x,u) \in \mathbb{R} \times \mathbb{R}^n | x \in J, u \in B\}$.
A function $u \in \mathscr{C}^1(J,\mathbb{R}^n)$ is said to be a solution of (1) PVI (1) if u satisfies the equation ${}^{C}D^{\alpha}u(x) = f(x,u(x))$a.e.$J$, and the condition $u(x_0) = x_0$.

2. PRELIMINARY CONCEPTS

2.1. Riemann–Liouville fractional integral :

Let $f : [a,b) \to \mathbb{R}$, be a continuous function and $\alpha \in \mathbb{R}_+$ the integral

$$\left(I_a^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\mathrm{d}t \tag{2}$$

is called a the Riemann-Liouville fractional integral of order α such that $a \in \mathbb{R}$ and $\Gamma(.)$ is the Gamma function. It should be noted that this definition exists for all $x \in [a,b]$. Also, it exists under more general conditions described in detail by Miller and Ross (1993).

2.2. Caputo fractional derivative

Let f be a function, such that $\frac{\mathrm{d}^n}{\mathrm{d}x^n} f$ is integrable, the fractional derivative of order α (with $n-1 \leqslant \alpha \leqslant n$) in the sense of Caputo is defined by :

$$\begin{aligned} {}^{C}D_a^{\alpha} f(x) &= \frac{1}{\Gamma(n-\alpha)} \int_a^t (x-t)^{n-\alpha-1} f^{(n)}(t)\mathrm{d}t \\ &= I_a^{n-\alpha}\left(\frac{\mathrm{d}^n}{\mathrm{d}x^n} f(x)\right). \end{aligned} \tag{3}$$

2.3. Definition of periodic solution

A non-constant solution $x(t)$ of any system is said to be a periodic solution if there exists $T > 0$ such that $x(t) = x(t+T)$, for all $t \in \mathbb{R}$ (Wiggins, 2003).

3. MAIN RESULTS

In 2008, Lakshmikantham and Vatsala [8] give the following existence result for IVP(1).

3.1. Existence results

Theorem 3.1 *[19] Assume that the function $f : E \to \mathbb{R}^n$ satisfies the following conditions of Carathéodory type :*

- *i) $f(x,u)$ is Lebesgue measurable with respect to x on J.*
- *ii) $f(x,u)$ is continuous with respect to u on B.*
- *iii) there exist a constant $\beta \in]0,\alpha[$ and a real-valued function $m(x) \in L^{\frac{1}{\beta}}(J)$ such that $\|f(x,u)\| \leqslant m(x)$, for almost every $x \in J$ and all $u \in B$.*

  Then, for $\alpha \in]0,1[$ there at least exists a solution of the IVP(1) on the interval $[x_0-h, x_0+h]$ where $h = \min\left\{ a, \left[\frac{b\Gamma(\alpha)}{M}\left(\frac{\alpha-\beta}{1-\beta}\right)^{1-\beta}\right]^{\frac{1}{\alpha-\beta}}\right\}$ and $M = \left(\int_{x_0}^{x_0+a} (m(t))^{\frac{1}{\beta}}\,\mathrm{d}t\right)^{\beta}$.

proof 3.1 *The IVP(1) for the case where $x \in [x_0, x_0+h]$ are only discussed. Similar approach could be used to verify the case where $x \in [x_0-h, x_0]$. We know that $f(x,u(x))$ is Lebesgue measurable in $[x_0,x_0+h]$ according to the conditions (i) and (ii).Direct calculation gives that $(x-t)^{\alpha-1} \in L^{\frac{1}{1-\beta}}[x_0,x]$, for $x \in [x_0,x_0+h]$.In light of the Hölder inequality, we obtain that, $(x-t)^{\alpha-1} f(t,u(t))$ is Lebesgue integrable with respect to $t \in [x_0,x]$ for all $x \in [x_0,x_0+h]$, and*

$$\int_{x_0}^{x} \|(x-t)^{\alpha-1} f(t,u(t))\|\mathrm{d}t \leqslant \left(\int_{x_0}^{x} ((x-t)^{\alpha-1})^{\frac{1}{1-\beta}}\mathrm{d}t\right)^{1-\beta} \left(\int_{x_0}^{x} (m(t))^{\frac{1}{\beta}}\mathrm{d}t\right)^{\beta}$$

for $u \in \mathscr{C}([x_0,x_0+h],\mathbb{R}^n)$, define norm of $\|u\|_* = \sup\limits_{t\in[x_0,x_0+h]} \|u(t)\|$.

Then $\mathscr{C}([x_0,x_0+h],\mathbb{R}^n)$ with $\|.\|_*$ is a Banach space.

Let $\Omega = \{u \in \mathscr{C}([x_0,x_0+h],\mathbb{R}^n) \ : \ \|u-u_0\|_* \leqslant b\}$. Then, Ω is a closed, bounded and convex subset of $\mathscr{C}([x_0,x_0+h],\mathbb{R}^n)$.

We now define a mapping on Ω as follows. For an element $u \in \Omega$, let

$$(Ru)(x) = u_0 + \frac{1}{\Gamma(\alpha)} \int_{x_0}^{x} (x-t)^{\alpha-1} f(t,u(t))\mathrm{d}t, \quad x \in [x_0,x_0+h].$$

a) *We shall show that for any* $u \in \Omega$, $Ru \in \Omega$.
In fact, by using the Hölder inequality and the condition (iii), we get

$$\begin{aligned} \|(Ru)(x)-u_0\| &\leqslant \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}(x-t)^{\alpha-1}m(t)\mathrm{d}t \\ &\leqslant \frac{1}{\Gamma(\alpha)}\left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta}(x-x_0)^{\alpha-\beta}\left(\int_{x_0}^{x_0+h}(m(t))^{\frac{1}{\beta}}\,\mathrm{d}t\right)^{\beta} \\ &\leqslant \frac{1}{\Gamma(\alpha)}\left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta}h^{\alpha-\beta}M \\ &\leqslant b, \quad pour\ x \in [x_0, x_0+h]. \end{aligned} \tag{4}$$

This means that $\|(Ru)(x)-u_0\|_* \leqslant b, \quad for\ x \in [x_0,x_0+h]$. *Hence, R is a mapping from* Ω *into itself.*

b) *We now show that R is completely continuous.*
First, we will show that Ris continuous. For any $u_m,\ u \in \Omega,\ m = 1,2,\dots$ *with* $\lim\limits_{m\to\infty}\|u_m - u\|_* = 0$, *we get*

$$\lim_{m\to\infty} u_m(x) = u(x),\ \ for\ x \in [x_0,x_0+h].$$

Thus, by the condition (ii), we have

$$\lim_{m\to\infty} f(x,u_m(x)) = f(x,u(x)),\ \ for\ x \in [x_0,x_0+h].$$

So, we can conclude that

$$\sup_{t\in[x_0,x_0+h]} \|f(t,u_m(t)) - f(t,u(t))\| \to 0,\ \ as\ m \to 0.$$

On the other hand,

$$\|(Ru_m)(x)-(Ru)(x)\| \leqslant \frac{h^{\alpha}}{\Gamma(\alpha+1)}\sup_{t\in[x_0,x_0+h]}\big\|f(t,u_m(t))-f(t,u(t))\big\|, \tag{5}$$

which implies

$$\|Ru_m - Ru\|_* \leqslant \frac{h^{\alpha}}{\Gamma(\alpha+1)}\sup_{t\in[x_0,x_0+h]}\big\|f(t,u_m(t))-f(t,u(t))\big\|. \tag{6}$$

Hence,

$$\|Ru_m - Ru\|_* \to 0 \quad as\ m \to \infty.$$

This means that R is continuous.
Next, we show $R\Omega$ *is relatively compact. It suffices to show that the family of functions* $\{Ru: \ \ u \in \Omega\}$ *is uniformly bounded and equicontinuous on* $[x_0,x_0+h]$.
For $u \in \Omega$, *we get*

$$\|Ru\|_* \leqslant \|u_0\| + b,$$

which means that $\{Ru: \ \ u \in \Omega\}$ *is uniformly bounded.*
On the other hand, for any $x_1,x_2 \in [x_0,x_0+h],\ \ x_1 < x_2$, *by using the Hölder inequality, we have*

$$\begin{aligned}
&\|(Ru)(x_2)-(Ru)(x_1)\| \\
&= \frac{1}{\Gamma(\alpha)}\left\|\int_{x_0}^{x_1}((x_2-t)^{\alpha-1}-(x_1-t)^{\alpha-1})f(t,u(t))\mathrm{d}t+\int_{x_1}^{x_2}(x_2-t)^{\alpha-1}f(t,u(t))\mathrm{d}t\right\| \\
&\leqslant \frac{1}{\Gamma(\alpha)}\int_{x_0}^{x_1}((x_1-t)^{\alpha-1}-(x_2-t)^{\alpha-1})m(t)\mathrm{d}t+\frac{1}{\Gamma(\alpha)}\int_{x_1}^{x_2}(x_2-t)^{\alpha-1}m(t)\mathrm{d}t \\
&\leqslant \frac{M}{\Gamma(\alpha)}\left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta}\left((x_1-x_0)^{\frac{\alpha-1}{1-\beta}+1}+(x_2-x_1)^{\frac{\alpha-1}{1-\beta}+1}-(x_2-x_0)^{\frac{\alpha-1}{1-\beta}+1}\right)^{1-\beta} \\
&+\frac{M}{\Gamma(\alpha)}\left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta}\left((x_2-x_1)^{\frac{\alpha-1}{1-\beta}+1}\right)^{1-\beta} \\
&\leqslant \frac{2M}{\Gamma(\alpha)}\left(\frac{1-\beta}{\alpha-\beta}\right)^{1-\beta}(x_2-x_1)^{\alpha-\beta}.
\end{aligned} \tag{7}$$

As $x_1 \to x_2$, the right-hand side of the above inequality tends to zero. Therefore $\{Ru: \ u\in\Omega\}$ is equicontinuous on $[x_0,x_0+h]$ and hence $R\Omega$ is relatively compact.

Bay Schauder's fixed point Theorem, there is a $u^*\in\Omega$ such that $Ru^*=u^*$, that is

$$u^*(x)=u_0+\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}(x-t)^{\alpha-1}f(t,u^*(t))\mathrm{d}t, \quad x\in[x_0,x_+h]. \tag{8}$$

It is easy to see that u^* is a solution of IVP (1) on $[x_0,x_0+h]$. This completes.

Corollary 3.1 *Let $f\in\mathscr{C}(J\times B,\mathbb{R}^n)$. Then, for $\alpha\in]0,1[$, there at least exists a solution of the IVP (1) on the interval $[x_0-h',x_0+h']$, where $h'=\min\left\{a,\left[\frac{b}{M'}\Gamma(\alpha+1)\right]^{\frac{1}{\alpha}}\right\}$ and $M'=\sup\limits_{(x,u)\in J\times B} f(x,u)$.*

proof 3.2 *The proof of **Corollaire 3.1** is similar to that of **Theorem 3.1**, it is therefore omitted.*

3.2. Uniqueness results

Theorem 3.2 *[19] Assume that there exist constant $\gamma\in]0,\alpha[$ and a real-valued function $\mu(x)\in L^{\frac{1}{\gamma}}[x_0,x_0+a]$ such that $\|f(x,u)-f(x,v)\|\leqslant\mu(x)\|u-v\|$, for almost every $x\in J$ and all $u,v\in B$. If the solutions of IVP (1) exist on $[x_0-h,x_0+h]$, then the solution of IVP (1), where $h<\min\left\{a,\left[\frac{\Gamma(\alpha)}{M_1}\left(\frac{\alpha-\gamma}{1-\gamma}\right)^{1-\gamma}\right]^{\frac{1}{\alpha-\gamma}}\right\}$ and $M_1=\left(\int_{x_0}^{x_0+a}(\mu(t))^{\frac{1}{\gamma}}\mathrm{d}t\right)^{\gamma}$.*

proof 3.3 *Assume that $v(x)$ and $u(x)$ are the solutions of IVP(1) on $[x_0,x_0+h]$.Then*

$$v(x)=u_0+\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}(x-t)^{\alpha-1}f(t,v(t))\mathrm{d}t, \quad x\in[x_0,x_0+h]$$

and

$$u(x)=u_0+\frac{1}{\Gamma(\alpha)}\int_{x_0}^{x}(x-t)^{\alpha-1}f(t,u(t))\mathrm{d}t, \quad x\in[x_0,x_0+h].$$

Thus according to the hypothesis of the theorem, we have

$$\|v(x)-u(x)\|\leqslant c\max_{t\in[x_0,x_0+\delta]}\|v(x)-u(x)\|,\ x\in[x_0,x_0+h],$$

where $c = \frac{M_1}{\Gamma(\alpha)}\left(\frac{1-\gamma}{\alpha-\gamma}\right)^{1-\gamma} h^{\alpha-\gamma} \in]0,1[$. *Therefore*

$$\max_{t\in[x_0,x_0+\delta]} \|v(x)-u(x)\| \leqslant c \max_{t\in[x_0,x_0+\delta]} \|v(x)-u(x)\|, \quad x\in[x_0,x_0+h].$$

Which implies

$$u(x)\equiv v(x), \quad x\in[x_0,x_0+h].$$

The proof is complete.

3.3. Existence results of periodicity of the long-time solution to fractional dynamical system

Tavazoei[14] claims that the existence of periodic solutions of IVP (1) is impossible. However, Yazdani and Salarieh [17] proved that this is not a general claim. Under certain conditions or circumstances, periodic solutions can exist in a fractional order system (1). The author proves that the periodic solution of IVP (1) can be tested by considering the steady-state behavior of the solution. Since the fractional derivative in IVP (1) is not local and different from the integer derivative, the solution of IVP (1) depends on its entire memory at any time t. In other words, if the difference between the lower end and the upper end of the fractional derivative of IVP (1) should be chosen to be large enough, that is, the lower end should be close to infinity, a periodic solution may be detected. After studying the periodicity of Weyl's fractional derivatives of periodic functions, this argument became stronger. In the section 19 of [6], the periodicity of fractional-order differential operator of periodic function has been discussed. We have the following results.

3.3.1. Memory dependency of solutions :

L'opérateur dérivé dans le calcul fractionnaire semble différent de celui d'ordre entier car il n'est pas local. En d'autres termes, lorsque cet opérateur apparaît dans une équation, la solution à l'instant t dépend de sa mémoire aux instants avant t. Pour montrer ce fait, considérons le théorème suivant.

Theorem 3.3 *[15] Let $f(x)$ satisfy the Lipschitz condition. Then, the solutions of the IVP (1) are memory dependent.*
It means that the solution of IVP (1), which is denoted by $\phi(t,x_a)$, and the solution of

$$\begin{cases} {}^{C}D_b^{\alpha} v = f(v), \\ y(b) = v_b \triangleq \phi(b,v_a), \quad b>a, \end{cases} \tag{9}$$

which is denoted by $\psi(t,v_b)$, do not coincide for $t \geqslant b$.

3.3.2. Existence of periodic solutions in steady state

Periodic orbits are steady state solutions; thus in a fractional order system due to the memory dependence of the derivatives, the whole memory of the solution should be considered in obtaining periodic orbits. It means that the difference between the lower terminal and the upper terminal in the derivatives should be chosen large enough. To detect periodic solution it implies that the lower terminal of the fractional derivative should approach to infinity. To prove this, let us consider the following theorem.

Theorem 3.4 *[17] The time-invariant fractional order systems presented by IVP (1) does not have any periodic solution unless the lower terminal of the derivative is $\pm\infty$.*

Exemple 3.1 *Consider the following fractional differential equation :*

$$ {}^{C}D_{a}^{\alpha}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \omega_0^{\alpha}\begin{bmatrix}\cos\frac{\alpha\pi}{2} & \sin\frac{\alpha\pi}{2}\\ -\sin\frac{\alpha\pi}{2} & \cos\frac{\alpha\pi}{2}\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} \tag{10}$$

where $0.5 < \alpha < 1$. *If the states are changed by the following transformation*

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0 & \sin\frac{\alpha\pi}{2}\\ -\omega_0^{\alpha} & \cos\frac{\alpha\pi}{2}\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}. \tag{11}$$

The differential equation(10)can be rewritten as

$$ {}^{C}D_{a}^{\alpha}\begin{bmatrix}y_1\\y_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\omega_0^{2\alpha} & 2\omega_0^{\alpha}\cos\frac{\alpha\pi}{2}\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}. \tag{12}$$

Thus this state space equation can be written as the following differential equation :

$$ {}^{C}D_{a}^{2\alpha}y_1 - 2\omega_0^{\alpha}\cos\frac{\alpha\pi}{2}\,{}^{C}D_{a}^{\alpha}y_1 + \omega_0^{2\alpha}y_1 = 0. \tag{13}$$

To show the periodicity of the answer of the Eq.(13), let us consider the analytical solution of Eq.(13) which is shown in Eq.(14). It should be noticed that this solution is obtained by taking Laplace transform from Eq.(13) and then applying the inverse Laplace transform to it.

$$y_1(t) = \frac{\sin_{\alpha,2-\alpha}(\omega_0(t-a))}{\omega_0^{\alpha}\sin\frac{\alpha\pi}{2}}\dot{y}_1(a) + \frac{\sin_{\alpha,1-\alpha}(\omega_0(t-a))}{\omega_0^{\alpha}\sin\frac{\alpha\pi}{2}}y_1(a) - \frac{2\omega_0^{\alpha}\cos\frac{\alpha\pi}{2}\sin_{\alpha,1}(\omega_0(t-a))}{\omega_0^{\alpha}\sin\frac{\alpha\pi}{2}}y_1(a) \tag{14}$$

where $\sin_{\alpha,\beta}(.)$ *is defined by :*

$$\sin_{\alpha,\beta}(z) \triangleq \frac{t^{\beta-1}}{2j}\left((E_{\alpha,\beta}(z^{\alpha}e^{j\frac{\alpha\pi}{2}}) - E_{\alpha,\beta}(z^{\alpha}e^{-j\frac{\alpha\pi}{2}})\right)$$

and $E_{\alpha,\beta}(.)$ *is generalized Mittag-Leffler function which is defined by Kilbas et al. (2006).*

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}, \ \ \alpha > 0, \ \beta > 0, \tag{15}$$

Although $\sin_{\alpha,\beta}(z)$ *is not periodic function for any finite value of* z, *when* z *tends to infinity, it shows periodic behavior, i.e.*

$$\lim_{z\to+\infty}\sin_{\alpha,\beta}(\omega_0 t) = \frac{\omega_0^{1-\beta}}{\alpha}\sin\left(\omega_0 t + \frac{(1-\beta)\pi}{2}\right). \tag{16}$$

Thus when lower terminal tend to infinity, Eq.(13) has periodic solution with the frequency of ω_0.

4. CONCLUSION AND DISCUSSION

In a fractional order system, when the lower end of the derivative operator tends to $-\infty$, that is, when the steady-state behavior of the solution is taken into account, periodic solutions may be detected. But these types of solutions cannot be found in fractional systems with any finite lower end. This has been proved in Theorem 3.4 and mentioned earlier by Tavazoei and Haeri (2009).

It has been shown that with this assumption, the harmonic balance method can be applied to fractional-order systems.

It can also be concluded that, like any other integer order system ; the existence of periodic orbits can be expected The chaotic attractor of the fractional order system, because chaos is the steady-state behavior of the dynamic system.

5. REFERENCES

[1] K. Diethelm and N.J. Ford, "Analysis of fractional differential equations," Journal of Mathematical Analysis and Applications, vol. 265, no. 2, pp. 229–248, 2002.

[2] A.Erdelyi , W.Magnus, Oberhettinger F and Tricomi F, Higher Transcendental Functions, Vol.III,Krieger Pub, Melbourne, Florida, (1981).

[3] A. Granas and J. Dugundji, Fixed Point Theory, Springer-Verlag, New York, 2003.

[4] M. Ishteva, L. Boyadjiev, R. Scherer, On the Caputo operator of fractional calculus and C-Laguerre functions, Mathematical Sciences Research Journal 9 (2005) 161–170.

[5] E. Kaslik, S. Sivasundaram, Non-existence of periodic solutions in fractional-order dynamical systems and a remarkable difference between integer and fractional-order derivatives of periodic functions [J]. Nonlinear Analysis : Real World Applications, 2012, 13(3) : 1489-1497.

[6] S.G. Samko , A.A. Kilbas and O.I. Marichev (1993), Fractional integrals and derivatives : theory and applications, Gordon and Breach, New York

[7] A.A. Kilbas, H.M. Srivastava and J.J. Trujillo, Theory and applications of fractional differential Equations, North-Holland Mathematical studies 204, Ed van Mill, Amsterdam, (2006).

[8] V. Lakshmikantham and A.S. Vatsala, Basic theory of fractional differential equations. Nonlinear Anal. 69 (2008), 2677-2682.

[9] Changpin Li, Deliang Qian and YangQuan Chen, On Riemann-Liouville and Caputo Derivatives, 2011.

[10] C.P. Li, F.H. Zeng, Numerical Methods for Fractional Differential Calculus [M]. Boca Raton,USA : Chapman and Hall/CRC, 2015.

[11] I. Podlubny, Fractional Differential Equations [M]. New York : Academic Press, 1999.

[12] Shahzad, Sarwar, Non-periodic solutions to fractional differential equations, vol.31, no.3, pp.250-264,2017.

[13] Y. Suansook , K. Paithoonwattanakij, Chaos in fractional order logistic model [C]//2009 Inter-national Conference on Signal Processing Systems, 2009 : 297-301.

[14] M. S. Tavazoei and M. Haeri, "A proof for non existence of periodic solutions in time invariant fractional order systems," Automatica, vol. 45, no. 8, pp. 1886–1890, 2009.

[15] M. Tavazoei, A note on fractional-order derivatives of periodic functions [J]. Automatica, 2010,46 : 945-948.

[16] S. Wiggins, Introduction to Applied Nonlinear Dynamical Systems and Chaos [M]. New York :Springer-Verlag, 2003.

[17] M. Yazdani, H. Salarieh, On the existence of periodic solutions in time-invariant fractional order systems [J]. Automatica, 2011, 47 : 1834-1837.

[18] E. Zeidler, Nonlinear Functional Analysis and Applications, Fixed Point Theorems Spring-Verlag, New York, 1986.

[19] Yong Zhou, EXISTENCE AND UNIQUENESS OF SOLUTIONS FOR A SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS, VOLUM 12, NUMBER 2(2009) :195-204.