PERIODIC NEGATIVE BINOMIAL INGARCH(1,1) MODEL

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ABSTRACT

In this paper, we introduce a class of periodic negative binomial integer-valued generalized autoregressive conditional heteroskedastic model. The basic probabilistic and statistical properties of this class are studied. Indeed, the first and the second moment periodically stationary conditions are established. The closed-forms of these moments are, under the obtained conditions, derived. Furthermore, the periodic autocovariance structure is also considered while providing the closed-form of the periodic autocorrelation function. The conditional maximum likelihood (CML) method is applied to estimate the underlying parameters. A simulation study and an application to a real data set are provided.

1. INTRODUCTION

As we know, the Poisson distribution is not always suitable for modeling and studying the integer time series, as was pointed out by Ristić et al. (2009). This is due to the equality of the mean and the variance which is not always verified in the real-world data set. In fact, many researchers have found it useful to propose other models that address the problem of overdispersion with another distribution and especially the negative binomial distribution. Indeed, Zhu (2011, 2012b) introduced a negative binomial INGARCH model to deal with the phenomena of overdispersion and subsequently, its zero-inflated version to treat the zero inflation situation. Some statistical properties and several extensions of this class have been taken by certain researchers, we can mention without claiming to be exhaustive the works of Xiong and Zhu (2019, 2021), Weiß et al. (2021), and many others. Furthermore, the generalized Poisson INGARCH model was presented and studied by Zhu (2012a), which can explain the overdispersion and underdispersion characteristics. Recently, Mao et al. (2020) proposed a more general mixture INGARCH model, which includes negative binomial and generalized Poisson mixture INGARCH models that can deal with multi-modality features either in the marginal or the conditional distribution.

However, despite the fact that much nonnegative integer-valued time series encountered in several fields as epidemiology, economic, environmental, criminology, and others reveal the periodicity feature in their autocovariance structures. Regardless of the various advantages and interesting properties satisfied by these models and more precisely by the negative binomial INGARCH model such as the positivity and the discreteness nature of the realizations, the volatility changes in time, this model is still unable to capture the periodicity feature, a feature that cannot be adequately accounted and described by time invariant parameter integer-valued time series models. This fact gave us a good reason and motivation to extend this class of time-invariant models to the periodic negative binomial INGARCH model with time-periodic coefficients as an extension of the periodic INGARCH(1,1) model introduced and studied by Bentarzi and Bentarzi (2017b). To our knowledge, the modeling of the periodically correlated, in the sense of Gladyshev (1961), integer-valued process has not received much attention in the last decade, and over time, the topic has made great progress, see, for example, Monteiro et al. (2015), Bentarzi and Bentarzi (2017a, 2017b), Bentarzi and Aries (2020a, 2020b) and most recently, Manaa and Bentarzi (2021a, 2021b), Prezotti et al. (2021) and among others.

2. NOTATIONS, DEFINITIONS AND MAIN ASSUMPTIONS

A periodically correlated integer-valued process, in the Gladyshev's sense (1961), $\{y_t;\, t \in \mathbb{Z}\}$ is said to satisfy a periodic negative binomial integer-valued generalized autoregressive conditional heteroskedastic model, with period S and orders p and q, noted in short by $\text{PNBINGARCH}_S\,(p,q)$, if it has the form :

$$\begin{aligned} y_t\big/ \mathscr{F}_{t-1} &\sim \mathscr{N}\mathscr{B}\,(r_t, p_t)\,, \\ \frac{1-p_t}{p_t} &= \lambda_t = \alpha_{0,t} + \textstyle\sum_{i=1}^{p} \alpha_{i,t} y_{t-i} + \sum_{j=1}^{q} \beta_{j,t} \lambda_{t-j}, \end{aligned} \tag{1}$$

where $\mathscr{F}_{t-1}$ denotes, as usually, the σ-field generated by $\{y_{t-1}, y_{t-2}, \ldots\}$ and r_t is a positive number. The parameters $\alpha_{i,t}$, $i = 0,1,...,p$, and $\beta_{j,t}$, $j = 1,...,q$, are periodic in t, with period S , i.e., $\alpha_{i,t+vS} = \alpha_{i,t}$, $\beta_{j,t+vS} = \beta_{j,t}$ and $r_{t+vS} = r_t$, $t, v \in \mathbb{Z}$. To avoid the possibility of zero or negative conditional variances, the following conditions for $\alpha_{i,t}$'s must be imposed : $\alpha_{0,t} > 0$, and $\alpha_{i,t} \geq 0$, $i = 1,...,p$ and $\beta_{j,t} \geq 0$, $j = 1,...,q$, $t \in \mathbb{Z}$. Particularly, we have, for $p = q = 1$, the periodic model, $\text{PNBINGARCH}_S\,(1,1)$, which is the goal of this paper :

$$\begin{aligned} y_t\big/ \mathscr{F}_{t-1} &\sim \mathscr{N}\mathscr{B}\,(r_t, p_t)\,, \\ \frac{1-p_t}{p_t} &= \lambda_t = \alpha_{0,t} + \alpha_{1,t} y_{t-1} + \beta_t \lambda_{t-1}, \end{aligned} \tag{2}$$

where, the parameters $\alpha_{i,t}$, $i = 0,1$, and β_t are periodic in t, with period S , i.e., $\alpha_{i,t+vS} = \alpha_{i,t}$, $i = 0,1$, $\beta_{t+vS} = \beta_t$ and $r_{t+vS} = r_t$, $t, v \in \mathbb{Z}$. Moreover, these parameters are such that : $\alpha_{0,t} > 0$, $\alpha_{1,t} \geq 0$, and $\beta_t \geq 0$, $t \in \mathbb{Z}$. Letting $t = s + \tau S$, $s = 1,2,...,S$ and $\tau \in \mathbb{Z}$, the last model can be rewritten in the equivalent form

$$\begin{aligned} y_{s+\tau S}\big/ \mathscr{F}_{s-1+\tau S} &\sim \mathscr{N}\mathscr{B}\,(r_s, p_{s+\tau S})\,, \\ \frac{1-p_{s+\tau S}}{p_{s+\tau S}} &= \lambda_{s+\tau S} = \alpha_{0,s} + \alpha_{1,s} y_{s-1+\tau S} + \beta_s \lambda_{s-1+\tau S}. \end{aligned} \tag{3}$$

3. STATIONARITY CONDITIONS

In this paragraph, we provide the conditions on parameters of the underlying integer-valued process to be periodically stationary in the first and second order. Furthermore, under these conditions, the closed forms of the periodic mean and the periodic variance are acquired.

3.1. Periodic stationarity in the first order

The results given in the following proposition establish the necessary and sufficient condition, for the process $\{y_t;\, t \in \mathbb{Z}\}$ satisfying (2) to be periodically stationary with respect to the first moment. The closed-forms of the periodic mean is then, under this condition, obtained.

Proposition 1 *The periodically correlated integer-valued process $\{y_t;\, t \in \mathbb{Z}\}$, satisfying the periodic negative binomial INGARCH$(1,1)$ model (2), is periodical stationary in mean, if and only if,*

$$\textstyle\prod_{i=1}^{S} \left(r_{i-1}\alpha_{1,i} + \beta_i\right) < 1.$$

Furthermore, the closed-form of the periodic mean $\mu_{y,s} = E\,(y_s) = r_s E\,(\lambda_s)$, $s = 1,...,S$, of such process is, under this condition, given by :

$$\mu_{y,s} = r_s \left(1 - \textstyle\prod_{i=1}^{S} \left(r_{i-1}\alpha_{1,i} + \beta_i\right)\right)^{-1} \textstyle\sum_{j=1}^{S} \left(\prod_{i=1}^{j-1} \left(r_{s-i}\alpha_{1,s-i+1} + \beta_{s-i+1}\right)\right) \alpha_{0,s-j+1},$$

with the convention $\prod_{i=1}^{j} x_i = 1$ if $j < 1$.

3.2. Periodic stationarity in the second order

The following proposition establishes a necessary and sufficient condition for the periodically integer-valued process $\{y_t;\ t \in \mathbb{Z}\}$ satisfying (2) to be stationary, with respect to the second order moment. The closed form of this moment is then, under this condition, obtained.

Proposition 2 *The periodically correlated integer-valued process $\{y_t;\ t \in \mathbb{Z}\}$ satisfying the periodic PNBINGARCH$(1,1)$ model (2) is periodically stationary in the second order if and only if*

$$\prod_{i=1}^{S}\left(r_{i-1}\alpha_{1,i}^2+\left(r_{i-1}\alpha_{1,i}+\beta_i\right)^2\right)<1. \tag{4}$$

Furthermore, the closed-form of the periodic variance $\gamma_y^{(s)}(0)=Var(y_s)$, $s=1,...,S$, of such process and the variance $\gamma_\lambda^{(s)}(0)=Var(\lambda_s)$ are, under this condition, given by

$$\begin{aligned}
\gamma_\lambda^{(s)}(0) &= \left(1-\left(\prod_{i=1}^{S}\psi_{2,i}\right)\right)^{-1}\sum_{j=0}^{S-1}\left(\prod_{i=1}^{j}\psi_{2,s-i+1}\right)F_{1,s-j+1},\\
\gamma_y^{(s)}(0) &= \mu_{y,s}+\left(r_s+r_s^2\right)\left(1-\left(\prod_{i=1}^{S}\psi_{2,i}\right)\right)^{-1}\sum_{j=0}^{S-1}\left(\prod_{i=1}^{j}\psi_{2,s-i+1}\right)F_{1,s-j+1}+\frac{1}{r_s}\mu_{y,s}^2,
\end{aligned}$$

where $\psi_{2,s}=\left(r_{s-1}\alpha_{1,s}^2+\left(r_{s-1}\alpha_{1,s}+\beta_s\right)^2\right)$, $F_{1,s}=\alpha_{1,s}^2\left(\mu_{y,s-1}+\frac{1}{r_{s-1}}\mu_{y,s-1}^2\right)$ *and* $\mu_{y,s}=r_s\mu_{\lambda,s}$ *is given in Proposition 1, with the convention* $\prod_{i=1}^{j}x_i=1$ *if* $j<1$.

4. AUTOCOVARIANCE STRUCTURE

The following proposition establishes the autocovariance structure of the process $\{y_t;\ t \in \mathbb{Z}\}$ satisfying the model (2).

Proposition 3 *The autocovariance structure of the periodically correlated integer-valued processes $\{y_t;\ t \in \mathbb{Z}\}$ and $\{\lambda_t;\ t \in \mathbb{Z}\}$ satisfying the model (2) are, under the condition (4), given as follows :*

$$\begin{aligned}
\gamma_y^{(s)}(0) &= \left(r_s+r_s^2\right)\left(1-\left(\prod_{i=1}^{S}\psi_{2,i}\right)\right)^{-1}\sum_{j=0}^{S-1}\left(\prod_{i=1}^{j}\psi_{2,s-i+1}\right)F_{1,s-j+1}+\frac{1}{r_s}\mu_{y,s}^2+\mu_{y,s},\\
\gamma_y^{(s)}(h) &= \left(\prod_{i=1}^{h-1}\psi_{1,s-i+1}\right)\left(\frac{\gamma_y^{(s-h)}(0)}{r_{s-h}}\left(\psi_{1,s-h+1}-\frac{\beta_{s-h+1}}{1+r_{s-h}}\right)+\frac{\beta_{s-h+1}}{1+r_{s-h}}\left(\frac{\mu_{y,s-h}^2}{r_{s-h}}+\mu_{y,s-h}\right)\right)r_{s-h+1},\\
\gamma_\lambda^{(s)}(0) &= \left(1-\left(\prod_{i=1}^{S}\psi_{2,i}\right)\right)^{-1}\sum_{j=0}^{S-1}\left(\prod_{i=1}^{j}\psi_{2,s-i+1}\right)F_{1,s-j+1},\\
\gamma_\lambda^{(s)}(h) &= \left(\prod_{i=1}^{h}\psi_{1,s-i+1}\right)\gamma_\lambda^{(s-h)}(0),
\end{aligned}$$

where $\psi_{2,s}=\left(r_{s-1}\alpha_{1,s}^2+\left(r_{s-1}\alpha_{1,s}+\beta_s\right)^2\right)$, $\psi_{1,s}=r_{s-1}\alpha_{1,s}+\beta_s$, $F_{1,s}=\alpha_{1,s}^2\left(\mu_{y,s-1}+\frac{1}{r_{s-1}}\mu_{y,s-1}^2\right)$ *with the convention* $\prod_{i=1}^{j}x_i=1$ *if* $j<1$.

Corollary 4 *The autocorrelation function of the periodically correlated integer-valued processes $\{y_t;\ t \in \mathbb{Z}\}$ and $\{\lambda_t;\ t \in \mathbb{Z}\}$ satisfying the model (2) are, under the condition (4), given as follows :*

$$\rho_y^{(s)}(\nu+kS)=\left(\prod_{i=1}^{S}\psi_{1,i}\right)^k\left(\prod_{i=1}^{\nu}\psi_{1,s-i+1}\right)\sqrt{\gamma_y^{(s-\nu)}(0)\Big/\gamma_y^{(s)}(0)}$$
$$\left(\frac{r_{s-\nu+1}}{r_{s-\nu}}\psi_{1,s-\nu+1}-\frac{r_{s-\nu+1}\beta_{s-\nu+1}}{r_{s-\nu}\left(1+r_{s-\nu}\right)}\left(1-\frac{\mu_{y,s-\nu}^2+r_{s-\nu}\mu_{y,s-\nu}}{\gamma_y^{(s-\nu)}(0)}\right)\right),\ \nu=1,...,S \text{ and } k\in\mathbb{N},$$

$$\rho_{\lambda}^{(s)}(\nu+kS)=\left(\prod_{i=1}^{S}\psi_{1,i}\right)^{k}\left(\prod_{i=1}^{\nu}\psi_{1,s-i+1}\right)\sqrt{\gamma_{\lambda}^{(s-\nu)}(0)\Big/\gamma_{\lambda}^{(s)}(0)},\ \nu=1,...,S \text{ and } k\in\mathbb{N}.$$

5. PARAMETER ESTIMATION

In the present section, we address the parameter estimation for our model given in (2.2), while considering the conditional maximum likelihood (CML) method.

5.1. Maximum likelihood estimation

Let the column vector of parameters $\underline{\theta}=\left(\underline{\alpha}_0',\underline{\alpha}_1',\underline{\beta}'\right)$, where the S-column vectors are $\underline{\alpha}_0'=\left(\alpha_{0,1},\alpha_{0,2},...,\alpha_{0,S}\right)'$, $\underline{\alpha}_1'=\left(\alpha_{1,1},\alpha_{1,2},...,\alpha_{1,S}\right)'$ and $\underline{\beta}'=\left(\beta_1,\beta_2,...,\beta_S\right)'$. For a simplicity in manipulation, we consider an observed time series of size $n=NS$, $\underline{y}_t=\left(y_1,y_2,...,y_n\right)$. Let $t=s+\tau S$, $s=1,2,...,S$ and $\tau=0,1,2,...,T-1$, then the conditional log-likelihood function can be written in the form :

$$\begin{aligned}\mathscr{L}\left(\underline{\theta}|\underline{y}_{s+\tau S}\right) &= \sum_{\tau=0}^{N-1}\sum_{s=1}^{S}\left[y_{s+\tau S}\log\left(\lambda_{s+\tau S}\right)-\left(r_s+y_{s+\tau S}\right)\log\left(1+\lambda_{s+\tau S}\right)-\log\left(y_{s+\tau S}!\right)\right.\\ &\left.+\sum_{i=1}^{y_{s+\tau S}}\log\left(i+r_s-1\right)\right].\end{aligned}$$

Analytical estimates of this log-likelihood function cannot be found, therefore numerical optimization methods must be employed.

6. SIMULATION RESULTS AND APPLICATION ON REAL DATA

In this section, we illustrate our obtained results and we assess the conditional maximum likelihood estimation on threes time series, for small, moderate, and relatively large sample sizes, and present a comparative study on real data, considering the Campylobacteriosis time series, studied by both Ferland *et al.* (2006) and Bentarzi and Bentarzi (2017*b*). This section will be detailed in the communication day.

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