STABILITY ANALYSIS FOR A GENERALIZED PROPORTIONAL FRACTIONAL LANGEVIN EQUATION WITH VARIABLE COEFFICIENT AND MIXED INTEGRO-DIFFERENTIAL BOUNDARY CONDITIONS

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ABSTRACT

In this paper, we study the Langevin equation within the generalized proportional fractional derivative. The proposed equation involves a variable coefficient and subjects to mixed integrodifferential boundary conditions. We introduce the generalized proportional fractional derivative and expose some of its features. We mainly investigate the existence, uniqueness and different types of Ulam stability of the solutions via fixed point theorems and inequality techniques. Finally, we provide an example to support our main results.

1. INTRODUCTION

In the last few decades, the investigation of fractional differential equations has been picking up much attention of researchers. This is due to the fact that fractional differential equations have various applications in engineering and scientific disciplines, for example, fluid dynamics, fractal theory, diffusion in porous media, fractional biological neurons, traffic flow, polymer rheology, neural network modeling, viscoelastic panel in supersonic gas flow, real system characterized by power laws, electrodynamics of complex medium, sandwich system identification, nonlinear oscillation of earthquake, models of population growth, mathematical modeling of the diffusion of discrete particles in a turbulent fluid, nuclear reactors and theory of population dynamics. The fractional differential equation is an important tool to describe the memory and hereditary properties of various materials and phenomena. The details on the theory and its applications may be found in books [19, 24, 27] and references therein.

Recently, fractional-order differential equations equipped with a variety of boundary conditions have been studied. The literature on the topic includes the existence and uniqueness results related to classical, initial value problem, periodic/anti-periodic, nonlocal, multi-point, integral boundary conditions, and Integral Fractional Boundary Condition, for instance, see [5, 6, 7, 8, 9, 10, 11, 12, **?**, 13, 15, 21, 22, 23, 29, 30].

The Langevin equation (first formulated by Langevin in 1908 to give an elaborate description of Brownian motion) is found to be an effective tool to describe the evolution of physical phenomena in fluctuating environments [20]. Although the existing literature on solutions of fractional Langevin equations is quite wide, can be found in [1, 2, 26, 30] and the references cited therein.

The aim of this paper is to study existence, uniqueness and different types of Ulam stability for the following generalized proportional fractional Langevin differential equation with variable coefficients and mixed intergo-differential boundary conditions

$$\begin{cases} {}^{C}\mathfrak{D}^{\beta,\rho}\left({}^{C}\mathfrak{D}^{\alpha,\rho}+\chi(t)\right)x(t)=\mathscr{F}\left(t,x(t)\right),t\in[0,1] \\ \\ x(0)=\delta, \quad x(1)=\theta I^{\mu,\rho}x(\xi), \end{cases} \tag{1}$$

where ${}^{C}\mathfrak{D}^{\nu,\rho}$ denote the generalized proportional Caputo and Riemann-Liouvill fractional derivatives of order $\nu \in \{\alpha,\beta\}$, respectively, $0<\alpha,\beta\leq 1, 1<\alpha+\beta\leq 2, \rho>0, I^{\mu,\rho}$ denotes the generalized proportional fractional integral of order $\mu>0,\rho>0,\chi\in C([0,1],\mathbb{R})$, the nonlinear function $\mathscr{F}\in C([0,1]\times\mathbb{R},\mathbb{R})$, the given constants $\delta,\theta\in\mathbb{R}$ and $\xi\in(0,1)$.

Here is a brief outline of the work. Section 2 provides the definitions and initial results presupposed to prove our key findings, and we make a present of an assistive lemma that extract the representation for the solutions of the problem (1). In Section 3, we establish the existence and uniqueness of the solutions taking advantage of the Banach fixed point theorem for the proposed problem, we also employ inequality techniques to prove the Ulam stability for the problem (1). The last section promotes our outcomes to the problem (1) by giving an illustrative example to support and justify the acquired results.

2. BASIC PRELIMINARIES

The notations and terminologies in this section are adopted from [18, 16, 28]. In control theory, a proportional derivative controller (PDC) for controller output u at time t with two tuning parameters has the algorithm

$$u(\mathrm{t})=\kappa_p E(\mathrm{t})+\kappa_d\frac{d}{d\mathrm{t}}E(\mathrm{t})$$

where κ_p is the proportional gain, κ_d is the derivative gain, and E is the input deviation or the error between the state variable and the process variable. The recent investigations have shown that PDC has direct incorporation in the control of complex networks models ; see [14] for more details.

For $\rho\in[0,1]$, let the functions $\kappa_0,\kappa_1:[0,1]\times\mathbb{R}\to[0,\infty)$ be continuous such that, for all $\mathrm{t}\in\mathbb{R}$

$$\lim_{\rho\to 0^+}\kappa_1(\rho,\mathrm{t})=1,\ \lim_{\rho\to 0^+}\kappa_0(\rho,\mathrm{t})=0,\ \lim_{\rho\to 1^-}\kappa_1(\rho,\mathrm{t})=0,\ \lim_{\rho\to 1^-}\kappa_0(\rho,\mathrm{t})=1$$

and $\kappa_1(\rho,\mathrm{t})\neq 0,\rho\in[0,1),\kappa_0(\rho,\mathrm{t})\neq 0,\rho\in(0,1]$. Then, Anderson and Ulness [3] defined the proportional derivative of order ρ by

$$\mathfrak{D}^{\rho}\xi(\mathrm{t})=\kappa_1(\rho,\mathrm{t})\xi(\mathrm{t})+\kappa_0(\rho,\mathrm{t})\xi'(\mathrm{t}) \tag{2}$$

provided that the right had side exists at $\mathrm{t}\in\mathbb{R}$ and $\xi':=\frac{d}{d\mathrm{t}}\xi$. For the operator given in (2), κ_1 is a type of proportional gain κ_p,κ_0 is a type of derivative gain κ_d,ξ is the error, and $u=\mathfrak{D}^{\rho}\xi$ is the controller output. The reader can refer to [4] for more details about the control theory of the proportional derivative and its component functions. We next restrict ourselves to the case that $\kappa_1(\rho,\mathrm{t})=1-\rho$ and $\kappa_0(\rho,\mathrm{t})=\rho$. Therefore, (2) becomes

$$\mathfrak{D}^{\rho}\xi(\mathrm{t})=(1-\rho)\xi(\mathrm{t})+\rho\xi'(\mathrm{t}) \tag{3}$$

It is easy to figure out that $\lim_{\rho\to 0^+}\mathfrak{D}^{\rho}\xi(\mathrm{t})=\xi(\mathrm{t})$ and $\lim_{\rho\to 1^-}\mathfrak{D}^{\rho}\xi(\mathrm{t})=\xi'(\mathrm{t})$. Thus, the derivative (3) is somehow considered to be more general than the conformable derivative, which evidently does not tend to the original functions as ρ tends to 0.

Definition 1 *[18] For $0<\rho\leq 1,\alpha\in\mathbb{C}$ and $\mathrm{Re}(\alpha)>0$, the GPF integral of $\mathscr{F}$ of order α is*

$$\left(I_a^{\alpha,\rho}\mathscr{F}\right)(\mathrm{t})=\int_a^{\mathrm{t}}\frac{(\mathrm{t}-s)^{\alpha-1}}{\rho^{\alpha}\Gamma(\alpha)}e^{\frac{\rho-1}{\rho}(\mathrm{t}-s)}\mathscr{F}(s)ds=\rho^{-\alpha}e^{\frac{\rho-1}{\rho}\mathrm{t}}I_a^{\alpha}\left(e^{\frac{1-\rho}{\rho}\mathrm{t}}\mathscr{F}(\mathrm{t})\right)$$

where I_a^{α} is Riemann-Liouville fractional integral.

Definition 2 *[18] For $0<\rho\leq 1, \alpha\in\mathbb{C}$ with $\mathrm{Re}(\alpha)\geq 0$, the GPF derivative of Caputo type of $\mathscr{F}$ of order α is*

$$\left({}_{c}\mathfrak{D}_a^{\alpha,\rho}\mathscr{F}\right)(\mathrm{t})=\int_a^{\mathrm{t}}\frac{(\mathrm{t}-s)^{n-\alpha-1}}{\rho^{n-\alpha}\Gamma(n-\alpha)}e^{\frac{\rho-1}{\rho}(\mathrm{t}-s)}\left(\mathfrak{D}^{n,\rho}\mathscr{F}\right)(s)ds$$

where $n=[\mathrm{Re}(\alpha)]+1$ and $[\mathrm{Re}(\alpha)]$ represents the integer part of the real number α.

Lemma 1 *[18] For $0<\rho\leq 1$ and $n=[\mathrm{Re}(\alpha)]+1$, we have $\left({}_{c}\mathfrak{D}_a^{\alpha,\rho}I_a^{\alpha,\rho}\mathscr{F}\right)(\mathrm{t})=\mathscr{F}(\mathrm{t})$, and*

$$\left(I_a^{\alpha,\rho}{}_{c}\mathfrak{D}_a^{\alpha,\rho}\mathscr{F}\right)(\mathrm{t})=\mathscr{F}(\mathrm{t})-\sum_{k=0}^{n-1}\frac{\left(\mathfrak{D}^{k,\rho}\mathscr{F}\right)(a)}{\rho^k k!}(\mathrm{t}-a)^k e^{\frac{\rho-1}{\rho}(\mathrm{t}-a)}$$

Proposition 2 *[18] Let $\alpha,\beta\in\mathbb{C}$ be such that $\mathrm{Re}(\alpha)\geq 0$ and $\mathrm{Re}(\beta)>0$. Then, for any $0<\rho\leq 1$ and $n=[\mathrm{Re}(\alpha)]+1$, we have*

- **(i)** $\left(I_a^{\alpha,\rho}e^{\frac{\rho-1}{\rho}\mathrm{t}}(\mathrm{t}-a)^{\beta-1}\right)(x)=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)\rho^\alpha}e^{\frac{\rho-1}{\rho}x}(x-a)^{\beta+\alpha-1},\quad \mathrm{Re}(\alpha)>0$
- **(ii)** $\left({}_{c}\mathfrak{D}_a^{\alpha,\rho}e^{\frac{\rho-1}{\rho}\mathrm{t}}(\mathrm{t}-a)^{\beta-1}\right)(x)=\frac{\rho^\alpha\Gamma(\beta)}{\Gamma(\beta-\alpha)}e^{\frac{\rho-1}{\rho}x}(x-a)^{\beta-\alpha-1},\quad \mathrm{Re}(\alpha)>n$
- **(iii)** $\left({}_{c}\mathfrak{D}_a^{\alpha,\rho}e^{\frac{\rho-1}{\rho}\mathrm{t}}(\mathrm{t}-a)^{k}\right)(x)=0,\quad \mathrm{Re}(\alpha)>n,\quad k=0,1,\ldots,n-1.$

In order to transform the main problem into a fixed point problem, (1) must be converted to an equivalent integral equation. To do this, we provide the following lemma.

Lemma 3 *Let $h:[0,1]\to\mathbb{R}$ be a continuous function, $0<\alpha,\beta,\rho\leq 1, 1<\alpha+\beta\leq 2$ and $\mu>0$. Then, the function $x\in C([0,1],\mathbb{R})$ is the solution to the following linear generalized proportional fractional Langevin equation equipped with mixed boundary conditions :*

$$\begin{cases} {}^{C}\mathfrak{D}^{\beta,\rho}\left({}^{C}\mathfrak{D}^{\alpha,\rho}+\chi(t)\right)x(t)=h(t), \quad t\in[0,1]\\ x(0)=\delta,\quad x(1)=\theta I^{\mu,\rho}x(\xi)\end{cases}\tag{4}$$

if and only if x satisfies the following fractional integral equation

$$\begin{aligned} &x(t)\\ =\ &\frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha+\beta-1}h(s)ds\\ -\ &\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\chi(s)x(s)ds+\delta e^{\frac{\rho-1}{\rho}t}\\ +\ &\frac{t^\alpha e^{\frac{\rho-1}{\rho}t}}{\Omega\rho^\alpha\Gamma(\alpha+1)}\left(\frac{\theta}{\rho^{\alpha+\beta+\mu}\Gamma(\alpha+\beta+\mu)}\int_0^\xi e^{\frac{\rho-1}{\rho}(\xi-s)}(\xi-s)^{\alpha+\beta+\mu-1}h(s)ds\right.\\ -\ &\frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_0^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\alpha+\beta-1}h(s)ds\\ -\ &\frac{\theta}{\rho^{\alpha+\mu}\Gamma(\alpha+\mu)}\int_0^\xi e^{\frac{\rho-1}{\rho}(\xi-s)}(\xi-s)^{\alpha+\mu-1}\chi(s)x(s)ds\\ &\left.+\frac{1}{\rho^\alpha\Gamma(\alpha)}\int_0^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\alpha-1}\chi(s)x(s)ds+\frac{\theta\delta\xi^\mu e^{\frac{\rho-1}{\rho}\xi}}{\rho^\mu\Gamma(\mu+1)}\right)\end{aligned}\tag{5}$$

where

$$\Omega:=\frac{e^{\frac{\rho-1}{\rho}}}{\rho^\alpha\Gamma(\alpha+1)}-\frac{\theta\xi^{\alpha+\mu}e^{\frac{\rho-1}{\rho}\xi}}{\rho^{\alpha+\mu}\Gamma(\alpha+\mu+1)}\neq 0\tag{6}$$

Proof. Let x be a solution of the problem (4). By using Lemma 1 with Proposition 2 (i), the first equation of (4) can be written as

$$x(t) = I^{\alpha+\beta,\rho}h(t) - I^{\alpha,\rho}\chi(t)x(t) + c_1\frac{t^{\alpha}e^{\frac{\rho-1}{\rho}t}}{\rho^{\alpha}\Gamma(\alpha+1)} + c_2e^{\frac{\rho-1}{\rho}t} \tag{7}$$

where $c_1, c_2 \in \mathbb{R}$. From the first condition of (4), $x(0) = \delta$, we obtain $c_2 = \delta$. Taking the operator $I^{\mu,\rho}$ into (7), we get

$$I^{\mu,\rho}_{x(t)} = I^{\alpha+\beta+\mu,\rho}h(t) - I^{\alpha+\mu,\rho}\chi(t)x(t) + c_1\frac{t^{\alpha+\mu}e^{\frac{\rho-1}{\rho}t}}{\rho^{\alpha+\mu}\Gamma(\alpha+\mu+1)} + \frac{\delta t^{\mu}e^{\frac{\rho-1}{\rho}t}}{\rho^{\mu}\Gamma(\mu+1)}$$

From the second condition of (4), $x(1) = \theta I^{\mu,\rho}x(\xi)$, which leads to

$$\begin{aligned} c_1 =& \frac{1}{\Omega}\Big(\theta I^{\alpha+\beta+\mu,\rho}h(\xi) - I^{\alpha+\beta,\rho}h(1) \\ & -\theta I^{\alpha+\mu,\rho}\chi(\xi)x(\xi) + I^{\alpha,\rho}\chi(1)x(1) + \frac{\theta\delta\xi^{\mu}e^{\frac{\rho-1}{\rho}\xi}}{\rho^{\mu}\Gamma(\mu+1)}\Big)\end{aligned}$$

where Ω is defined by (6). Substituting the values of c_1 and c_2 into (7), we get the fractional integral equation (5). Conversely, it is easily to show by direct computation that the solution $x(t)$ is given by (5) satisfies the problem (4) under the given conditions. The proof is completed. ■ For the completeness, we recall the following tools [29].

Theorem 4 *Let $\mathfrak{B}$ be a non-empty closed subset of a Banach space E. Then any contraction mapping T from $\mathfrak{B}$ into itself has a unique fixed point.*

Theorem 5 *Let E be a Banach space and $T : E \to E$ is a completely continuous operator and the set $\mathfrak{F} = \{x \in E : x = \rho Tx, 0 < \rho \leq 1\}$ is bounded. Then T has a fixed point in E.*

3. MAIN RESULTS

In this section, we present the existence and uniqueness of solutions to problem (1) based on the Banach's fixed point theorem.

Let $\mathscr{C} = C([0,1],\mathbb{R})$ be the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ equipped with the norm $\|x\| = \max_{t\in[0,1]}\{|x(t)|\}$.

In view of Lemma 3, an operator $Q : \mathscr{C} \to \mathscr{C}$ is defined by

$$\begin{aligned} & Qx(t) \\ = & \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha+\beta-1}h(s)ds \\ - & \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\chi(s)x(s)ds + \delta e^{\frac{\rho-1}{\rho}t} \\ + & \frac{t^{\alpha}e^{\frac{\rho-1}{\rho}t}}{\Omega\rho^{\alpha}\Gamma(\alpha+1)}\Big(\frac{\theta}{\rho^{\alpha+\beta+\mu}\Gamma(\alpha+\beta+\mu)}\int_0^{\xi} e^{\frac{\rho-1}{\rho}(\xi-s)}(\xi-s)^{\alpha+\beta+\mu-1}h(s)ds \\ - & \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_0^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\alpha+\beta-1}h(s)ds \\ - & \frac{\theta}{\rho^{\alpha+\mu}\Gamma(\alpha+\mu)}\int_0^{\xi} e^{\frac{\rho-1}{\rho}(\xi-s)}(\xi-s)^{\alpha+\mu-1}\chi(s)x(s)ds \\ & + \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_0^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\alpha-1}\chi(s)x(s)ds + \frac{\theta\delta\xi^{\mu}e^{\frac{\rho-1}{\rho}\xi}}{\rho^{\mu}\Gamma(\mu+1)}\Big) \end{aligned} \tag{8}$$

where Ω is defined by (6).

For the sake of computational convenience, we make use of the following constants :

$$\chi^* = \sup_{t\in[0,1]} |\chi(t)|$$
$$\Lambda_1 = \frac{1}{|\Omega|\rho^{2\alpha+\beta}\Gamma(\alpha+1)}\left(\frac{|\Omega|\rho^{\alpha}\Gamma(\alpha+1)+1}{\Gamma(\alpha+\beta+1)} + \frac{|\theta|\xi^{\alpha+\beta+\mu}}{\rho^{\mu}\Gamma(\alpha+\beta+\mu+1)}\right) \tag{9}$$
$$\Lambda_2 = \frac{1}{|\Omega|\rho^{\alpha}\Gamma(\alpha+1)}\left((|\Omega|\rho^{\alpha}\Gamma(\alpha+1)+1)\, I^{\alpha,\rho}|\chi(1)| + |\theta| I^{\alpha+\mu,\rho}|\chi(\xi)|\right)$$

We prove our main results under the following assumptions :
(H_1) the function $\mathscr{F} : [0,1]\times\mathbb{R}\to\mathbb{R}$ is continuous,
(H_2) there exist non-negative continuous functions $a_1 \in \mathscr{C}$ such that

$$|\mathscr{F}(t,x_1) - \mathscr{F}(t,x_2)| \le a_1(t)\,|x_1 - y_1|\,, \forall x_1,x_2\in\mathbb{R}, t\in[0,1]$$

with $a_1^* = \sup_{t\in[0,1]} a_1(t)$.

3.1. Existence and uniqueness results based on the Banach fixed point theorem

The existence and uniqueness results of a solution for problem (1) will be proved by using Banach fixed point theorem.

Theorem 6 *Assume that (H_1) and (H_2) are satisfied. If $\left[\left(a_1^*\Lambda_1 + \Lambda_2\right)\right] < 1$, then problem (1) has a unique solution in $\mathscr{C}$, where $\Lambda_i (i = 1,2)$ are defined in (9).*

Proof. Show that $Q : \mathscr{C}\to\mathscr{C}$ is contraction. For any $x, y \in \mathscr{C}$ and for each $t\in[0,1]$, we have

$$\begin{aligned}
&|(Qx)(t) - (Qy)(t)| \\
&\le \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha+\beta-1}\,|\mathscr{F}(s,x(s)) - \mathscr{F}(s,y(s))|\,ds \\
&+ \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}|\chi(s)||x(s)-y(s)|ds \\
&+ \frac{t^{\alpha}e^{\frac{\rho-1}{\rho}t}}{|\Omega|\rho^{\alpha}\Gamma(\alpha+1)}\left(\frac{|\theta|}{\rho^{\alpha+\beta+\mu}\Gamma(\alpha+\beta+\mu)}\int_0^{\xi} e^{\frac{\rho-1}{\rho}(\xi-s)}(\xi-s)^{\alpha+\beta+\mu-1}\mid\mathscr{F}(s,x(s))\right. \\
&- \mathscr{F}(s,y(s))\mid ds \\
&+ \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_0^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\alpha+\beta-1}\,|\mathscr{F}(s,x(s)) - \mathscr{F}(s,y(s))|\,ds \\
&+ \frac{|\theta|}{\rho^{\alpha+\mu}\Gamma(\alpha+\mu)}\int_0^{\xi} e^{\frac{\rho-1}{\rho}(\xi-s)}(\xi-s)^{\alpha+\mu-1}|\chi(s)||x(s)-y(s)|ds \\
&\left.+\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_0^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\alpha-1}|\chi(s)||x(s)-y(s)|ds\right)
\end{aligned}$$

Using the property of $e^{\frac{\rho-1}{\rho}(u-s)} \leq 1$ for $0 \leq s < u < t \leq 1$ and (H_2) implies that

$$\begin{aligned}
&|(Qx)(t)-(Qy)(t)| \\
&\leq \frac{1}{|\Omega|\rho^{2\alpha+\beta}\Gamma(\alpha+\beta)}\left(|\Omega|\rho^{\alpha}+\frac{1}{\Gamma(\alpha+1)}\right) \\
&\times \int_0^1 (1-s)^{\alpha+\beta-1}\left(a_1(s)|x(s)-y(s)|\right)ds \\
&+ \frac{|\theta|}{|\Omega|\rho^{2\alpha+\beta+\mu}\Gamma(\alpha+1)\Gamma(\alpha+\beta+\mu)} \\
&\times \int_0^{\xi} (\xi-s)^{\alpha+\beta+\mu-1}\left(a_1(s)|x(s)-y(s)|\right)ds \\
&+ \frac{1}{|\Omega|\rho^{2\alpha}\Gamma(\alpha)}\left(|\Omega|\rho^{\alpha}+\frac{1}{\Gamma(\alpha+1)}\right)\int_0^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\alpha-1}|\chi(s)||x(s)-y(s)|ds \\
&+ \frac{|\theta|}{|\Omega|\rho^{2\alpha+\mu}\Gamma(\alpha+1)\Gamma(\alpha+\mu)}\int_0^{\xi} e^{\frac{\rho-1}{\rho}(\xi-s)}(\xi-s)^{\alpha+\mu-1}|\chi(s)||x(s)-y(s)|ds
\end{aligned}$$

Then

$$\begin{aligned}
&\|Qx-Qy\| \qquad (10)\\
&\leq \Bigg\{\frac{a_1^*}{|\Omega|\rho^{2\alpha+\beta}\Gamma(\alpha+1)}\left(\frac{|\Omega|\rho^{\alpha}\Gamma(\alpha+1)+1}{\Gamma(\alpha+\beta+1)}+\frac{|\theta|\xi^{\alpha+\beta+\mu}}{\rho^{\mu}\Gamma(\alpha+\beta+\mu+1)}\right) \\
&+ \frac{1}{|\Omega|\rho^{\alpha}\Gamma(\alpha+1)}\left((|\Omega|\rho^{\alpha}\Gamma(\alpha+1)+1)I^{\alpha,\rho}|\chi(1)|+|\theta|I^{\alpha+\mu,\rho}|\chi(\xi)|\right)\Bigg\}\|x-y\| \\
&= \left(a_1^*\Lambda_1+\Lambda_2\right)\|x-y\| \qquad (11)
\end{aligned}$$

It follows from (8) and (10) that $\|Qx-Qy\| \leq \left[\left(a_1^*\Lambda_1+\Lambda_2\right)\right]\|x-y\|$, where $\Lambda_i(i=1,2)$ are defined in (9). Hence, by Banach fixed point theorem, Q is a contraction. Therefore, it has a unique fixed point, that is, the unique solution of problem (1). ■

3.2. Stability Results

In the recent section, we interested to studied UH and UHR stability of System (1).

Definition 3 *System (1) is UH stable if there exists a real number* $c_{\mathscr{F}} > 0$ *such that, for each* $\varepsilon \in \mathbb{R}^+$ *and for each* $x \in \mathscr{C}$ *satisfying*

$$\begin{cases} \left|{}^{C}\mathfrak{D}^{\beta,\rho}\left({}^{C}\mathfrak{D}^{\alpha,\rho}+\chi(t)\right)x(t)-\mathscr{F}(t,x(t))\right| \leq \varepsilon, \quad (\mathfrak{t} \in I), \\ x(0)=\delta, x(1)=\theta I^{\mu,\rho}x(\xi), \end{cases} \tag{12}$$

there exists a unique solution $\tilde{x} \in \mathscr{C}$ *of (1) with*

$$\|x-\tilde{x}\| \leq c_{\mathscr{F}}\varepsilon.$$

Definition 4 *System (1) is generalized UH stable (GUH) if there exists* $C_{\mathscr{F}} \in C\left(\mathbb{R}^+,\mathbb{R}^+\right), C_{\mathscr{F}}(0) = 0$ *such that for each* $\varepsilon \in \mathbb{R}^+$ *and for each* $x \in \mathscr{C}$ *satisfying* (12), *there exists a unique solution* $\tilde{x} \in \mathscr{C}$ *of (1) with*

$$\|x-\tilde{x}\| \leq C_{\mathscr{F}}(\varepsilon).$$

Remark 1 *A function $\tilde{x} \in \mathscr{C}$ is a solution of inequality (3) if and only if there exists a function $\mathscr{F} \in \mathscr{C}$ (which depends on solution $\tilde{x}$) such that*
1. $|\mathscr{G}(\mathrm{t})| \leq \varepsilon, \mathrm{t} \in \mathrm{I}.$
2. ${}^{C}\mathfrak{D}^{\beta,\rho}\left({}^{C}\mathfrak{D}^{\alpha,\rho}+\chi(t)\right)x(t)=\mathscr{F}\left(t,x(t)\right))+\mathscr{G}(\mathrm{t}), \quad \mathrm{t} \in \mathrm{I}.$

Now, we discuss the UH stability of solution to the problem (1).

Theorem 7 *Suppose that the conditions (H_2) and $\left[\left(a_1^*\Lambda_1+\Lambda_2\right)\right]<1$ are fulfilled. Then, the solution of (1) is UH and GUH stable.*

Proof. Let $\varepsilon > 0$ and let $\tilde{x} \in \mathscr{C}$ be a function which satisfies the inequality (12) and let $x \in \mathscr{C}$ the unique solution of the following problem

$$\begin{cases} {}^{C}\mathfrak{D}^{\beta,\rho}\left({}^{C}\mathfrak{D}^{\alpha,\rho}+\chi(t)\right)x(t)=\mathscr{F}\left(t,x(t)\right), \quad (\mathrm{t} \in I), \\ \\ x(0)=\delta, x(1)=\theta I^{\mu,\rho}x(\xi), \end{cases} \tag{13}$$

By Lemma 3, we have

$$\begin{aligned} x(t) &= \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha+\beta-1}\mathscr{F}\left(s,x(s)\right)ds \\ &- \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\chi(s)x(s)ds+\delta e^{\frac{\rho-1}{\rho}t} \\ &+ \frac{t^{\alpha}e^{\frac{\rho-1}{\rho}t}}{\Omega\rho^{\alpha}\Gamma(\alpha+1)}\left(\frac{\theta}{\rho^{\alpha+\beta+\mu}\Gamma(\alpha+\beta+\mu)}\int_0^{\xi} e^{\frac{\rho-1}{\rho}(\xi-s)}(\xi-s)^{\alpha+\beta+\mu-1}\mathscr{F}\left(s,x(s)\right)ds\right. \\ &- \frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_0^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\alpha+\beta-1}\mathscr{F}\left(s,x(s)\right)ds \\ &- \frac{\theta}{\rho^{\alpha+\mu}\Gamma(\alpha+\mu)}\int_0^{\xi} e^{\frac{\rho-1}{\rho}(\xi-s)}(\xi-s)^{\alpha+\mu-1}\chi(s)x(s)ds \\ &\left.+ \frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_0^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\alpha-1}\chi(s)x(s)ds+\frac{\theta\delta\xi^{\mu}e^{\frac{\rho-1}{\rho}\xi}}{\rho^{\mu}\Gamma(\mu+1)}\right) \end{aligned}$$

Since we have assumed that $\tilde{x}$ is a solution of (3), hence we have by Remark 1.

$$\begin{cases} {}^{C}\mathfrak{D}^{\beta,\rho}\left({}^{C}\mathfrak{D}^{\alpha,\rho}+\chi(t)\right)\tilde{x}(t)=\mathscr{F}\left(t,\tilde{x}(t)\right), \quad (\mathrm{t} \in I), \\ \\ \tilde{x}(0)=\delta, \tilde{x}(1)=\theta I^{\mu,\rho}\tilde{x}(\xi), \end{cases} \tag{14}$$

Again by Lemma 3, we have

$$\begin{aligned}
\tilde{x}(t) = &\frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha+\beta-1}\mathscr{F}(s,x(s))\,ds \\
&-\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\chi(s)\tilde{x}(s)ds+\delta e^{\frac{\rho-1}{\rho}t} \\
&+\frac{t^{\alpha}e^{\frac{\rho-1}{\rho}t}}{\Omega\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\frac{\theta}{\rho^{\alpha+\beta+\mu}\Gamma(\alpha+\beta+\mu)}\int_0^{\xi} e^{\frac{\rho-1}{\rho}(\xi-s)}(\xi-s)^{\alpha+\beta+\mu-1}\mathscr{F}(s,x(s))\,ds \\
&-\frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_0^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\alpha+\beta-1}\mathscr{F}(s,x(s))\,ds \\
&-\frac{\theta}{\rho^{\alpha+\mu}\Gamma(\alpha+\mu)}\int_0^{\xi} e^{\frac{\rho-1}{\rho}(\xi-s)}(\xi-s)^{\alpha+\mu-1}\chi(s)\tilde{x}(s)ds \\
&+\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_0^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\alpha-1}\chi(s)\tilde{x}(s)ds+\frac{\theta\delta\xi^{\mu}e^{\frac{\rho-1}{\rho}\xi}}{\rho^{\mu}\Gamma(\mu+1)}\Bigg) \\
&+\frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha+\beta-1}\mathscr{G}(s)ds \\
&+\frac{t^{\alpha}e^{\frac{\rho-1}{\rho}t}}{\Omega\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\frac{\theta}{\rho^{\alpha+\beta+\mu}\Gamma(\alpha+\beta+\mu)}\int_0^{\xi} e^{\frac{\rho-1}{\rho}(\xi-s)}(\xi-s)^{\alpha+\beta+\mu-1}\mathscr{G}(s)ds \\
&-\frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_0^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\alpha+\beta-1}\mathscr{G}(s)ds\Bigg)
\end{aligned}$$

On the other hand, we have, for each $\varsigma \in \mathrm{I}$

$$\begin{aligned}
&|\tilde{x}(t)-x(t)| \\
&=\frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha+\beta-1}\left|\mathscr{F}(s,\widetilde{x}(s))-\mathscr{F}(s,x(s))\right|ds \\
&-\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha-1}\chi(s)\left|\widetilde{x}(s)-x(s)\right|ds+\delta e^{\frac{\rho-1}{\rho}t} \\
&+\frac{t^{\alpha}e^{\frac{\rho-1}{\rho}t}}{\Omega\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\frac{\theta}{\rho^{\alpha+\beta+\mu}\Gamma(\alpha+\beta+\mu)}\int_0^{\xi} e^{\frac{\rho-1}{\rho}(\xi-s)}(\xi-s)^{\alpha+\beta+\mu-1}\left|\mathscr{F}(s,\widetilde{x}(s))-\mathscr{F}(s,x(s))\right|ds \\
&-\frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_0^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\alpha+\beta-1}\left|\mathscr{F}(s,\widetilde{x}(s))-\mathscr{F}(s,x(s))\right|ds \\
&-\frac{\theta}{\rho^{\alpha+\mu}\Gamma(\alpha+\mu)}\int_0^{\xi} e^{\frac{\rho-1}{\rho}(\xi-s)}(\xi-s)^{\alpha+\mu-1}\chi(s)\left|\widetilde{x}(s)-x(s)\right|ds \\
&+\frac{1}{\rho^{\alpha}\Gamma(\alpha)}\int_0^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\alpha-1}\chi(s)\left|\widetilde{x}(s)-x(s)\right|ds+\frac{\theta\delta\xi^{\mu}e^{\frac{\rho-1}{\rho}\xi}}{\rho^{\mu}\Gamma(\mu+1)}\Bigg) \\
&+\frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_0^t e^{\frac{\rho-1}{\rho}(t-s)}(t-s)^{\alpha+\beta-1}\mathscr{G}(s)ds \\
&+\frac{t^{\alpha}e^{\frac{\rho-1}{\rho}t}}{\Omega\rho^{\alpha}\Gamma(\alpha+1)}\Bigg(\frac{\theta}{\rho^{\alpha+\beta+\mu}\Gamma(\alpha+\beta+\mu)}\int_0^{\xi} e^{\frac{\rho-1}{\rho}(\xi-s)}(\xi-s)^{\alpha+\beta+\mu-1}\mathscr{G}(s)ds \\
&-\frac{1}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta)}\int_0^1 e^{\frac{\rho-1}{\rho}(1-s)}(1-s)^{\alpha+\beta-1}\mathscr{G}(s)ds\Bigg)
\end{aligned}$$

Hence using part 1 of Remark 1 and (H_2) we can get

$$|\tilde{x}-x| \leq \Lambda_1 \varepsilon + [(a_1^* \Lambda_1 + \Lambda_2)] \|x-\tilde{x}\|,$$

In consequence, it follows that

$$\|\tilde{x}-x| \leq \frac{\Lambda_1}{1-\left[\left(a_1^* \Lambda_1 + \Lambda_2\right)\right]} \varepsilon.$$

If we let $c_{\mathscr{F}} = \frac{\Lambda_1}{1-[(a_1^*\Lambda_1+\Lambda_2)]}$, then, the UH stability condition is satisfied. More generally, for $C_{\mathscr{F}}(\varepsilon) = \frac{\Lambda_1}{1-[(a_1^*\Lambda_1+\Lambda_2)]}\varepsilon$; $C_{\mathscr{F}}(0) = 0$ the generalized UH stability condition is also satisfied. This completes the proof. ■

4. EXAMPLE

Example 1 *Consider the following problem*

$$\begin{cases} {}^{C}\mathfrak{D}^{0.5,0.9}\left(\mathfrak{D}^{0.7,0.9} + \frac{1}{16} t^2 e^{\frac{\rho-1}{\rho} t}\right) x(t) = \mathscr{F}(t, x(t)), \quad t \in [0,1] \\ x(0) = 0.2, \quad x(1) = 5 I^{0.1,0.9} x(0.8) \end{cases} \tag{15}$$

Here $\alpha = 0.7, \beta = 0.5, \mu = 0.1, \rho = 0.9, \delta = 0.2, \theta = 5$ *and* $\xi = 0.8$. *From the given data, we can utilize Maple to obtain that* $\Omega \approx -2.0311 \neq 0, \Lambda_1 \approx 1.2965$, $\Lambda_2 \approx 0.4131$ *and* $\bar{\chi} = \sup_{t \in [0,1]} |\chi(t)| \approx 0.0559$. *Let* $\mathscr{F}$ *be defined by*

$$\mathscr{F}(t,x,y) = 2t^3 + \frac{9e^{-t}}{10(t+5)} \cdot \frac{|x|}{4+|x|}.$$

For $x_1, x_2 \in \mathbb{R}$, *we have*

$$|\mathscr{F}(t,x_1) - \mathscr{F}(t,x_2)| \leq \frac{9e^{-t}}{40(t+5)} |x_1 - x_2|$$

Then we have $a_1(t) = \frac{9e^{-t}}{40(t+5)}$ *with* $a_1^* = 0.045$. *Hence*

$$(a_1^* \Lambda_1 + \Lambda_2) \approx 0.3705112437 < 1$$

This ensures the existence of unique solution for system (15) according to Theorem 6. Furthermore, we can compute

$$\frac{\Lambda_1}{\left[1-\left(a_1^*\Lambda_1+\Lambda_2\right)\right]} \approx 5.052686151 > 0$$

Thus, by use of Theorem 7, (15) is UH stable and consequently generalized UH stable.

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