# ABSTRACT NONLINEAR BOUNDARY IMPLICIT CAPUTO–EXPONENTIAL TYPE FRACTIONAL DIFFERENTIAL EQUATIONS

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**ABSTRACT**

In this paper, we establish sufficient conditions for the existence of solutions and Ulam-Hyers-Rasias stability for a class of boundary value problem for implicit fractional differential equations with Caputo–Exponential fractional derivative in Banach space. The arguments are based upon the Darbo's theorem fixed point and Mönch's fixed point theorem together with the measure of noncompactness. An example is included to show the applicability of our results.
**Key words and phrases :** Caputo-Exponential type fractional derivative, fractional differential equations, Boundary value conditions, existence, fixed point, Ulam stability.
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## 1. INTRODUCTION

The fractional calculus play a very important role in numerous application as in industrial robotics, in optimal control, in population dynamics, etc. That is why many authors are studying and developing the theory of fractional calculus and fractional differential equations and their important properties. See, for example, the books [1, 4, 5, 7, 8, 19, 27] and the research papers [2, 3, 12–16, 23–25] and the references therein.

In [10] by means of the Banach contraction principle, Benchohra and Bouriah studied the existence and Ulam stability of nonlinear fractional boundary value problem involving Caputo derivative

$$^{c}D_{0}^{\alpha}y(t) = \chi(t, y(t),\ ^{c}D_{0}^{\alpha}y(t)), \text{ for each, } t \in J := [0,T], T > 0,\ 0 < \alpha \leq 1,$$

$$d_1 y(0) + d_2 y(T) = d_3,$$

and

$$^{c}D_{0}^{\alpha}y(t) = \chi(t, y(t),\ ^{c}D_{0}^{\alpha}y(t)), \text{ for each, } t \in J,\ 0 < \alpha \leq 1,$$
$$y(0) + \vartheta(y) = y^{*},$$

where $\chi : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $\vartheta : C(J,\mathbb{R}) \to \mathbb{R}$ are a given functions and $y^{*}, d_1, d_2, d_3 \in \mathbb{R}$. And in [11] by means of technique of measure of noncompactness and the fixed point theorems of Darbo and Mönch, the authors studied the existence of nonlinear fractional boundary value problem involving Caputo derivative

$$^{c}D_{0}^{\rho}x(t) = \chi(t, x(t),\ ^{c}D_{0}^{\rho}x(t)), \text{ for each, } t \in J := [0,b], b > 0,\ 0 < \rho \leq 1,$$

$$d_1 x(0) + d_2 y(b) = d_3,$$

and

$$^{c}D_{0}^{\rho}x(t) = \chi(t, x(t),\ ^{c}D_{0}^{\rho}x(t)), \text{ for each, } t \in J, b > 0,\ 0 < \rho \leq 1,$$

$$x(0)+\vartheta(x)=x^*,$$

where $\chi : J\times E\times E\to E$, $\vartheta : C(J,E)\to E$ are a given functions and $d_1,d_2\in\mathbb{R}$, $d_3,x^*\in E$, and $(E,\|\cdot\|)$ is a real Banach space.

In [19] (p.99 sect. 2.5), Kilbas *et al.* present the definitions and some properties of the fractional integral and fractional derivatives of a function $f$ with respect to another function $\psi$. In [26], Tariboon and Ntouyas get $\psi(t)=e^t$ and introduce a new class of exponential type fractional integral and exponential type fractional derivative. Meanwhile, Malti *et al.* [20] establish the existence and uniqueness results of solutions for a class of impulsive boundary value problem for nonlinear implicit fractional differential equations involving Caputo exponential type fractional derivative.

Motivated by the above works, we investigate the existence of solutions and Ulam-Hyers-Rasias stability (**U-H-R**) for a class of the boundary value problem (**BVP**) for the following nonlinear implicit fractional-order differential equation (**NIFDE**) in Banach space :

$$ {}^{e}_{c}D^{\rho}_{0}\omega(t)=f(t,y(t),\, {}^{e}_{c}D^{\rho}_{0}\omega(t)),\ \text{for each},\ t\in J:=[0,b], b>0,\ 0<\rho\leq 1, \tag{1}$$

$$c_1\omega(0)+c_2\omega(b)=\delta, \tag{2}$$

where ${}^{e}_{c}D^{\rho}_{0}$ is the exponential left-sided of Caputo–Exponential type fractional derivative, $f : J\times E\times E\to E$ is a given function and $c_1$, $c_2$, are real constants with $c_1+c_2\neq 0$, and $\delta\in E$, where $(E,\|\cdot\|)$ is a real Banach space.

The paper is organized as follows. In Sect. 2, we introduce Some notations, definitions, lemmas and theorems. In the first subsection of Sect. 3, we prove existence results of the BVP for NIFDE (1)-(2) by using Darbo's fixed point theorem and on Mönch's fixed point theorem combined with the measure of noncompactness. While in the Sect. 4, we study the U-H-R stability. In addition to illustrate the results presented, we give an example.

## 2. PRELIMINARIES

In this section, we introduce some notations, definitions, lemmas, properties and fixed point theorems that will be used in the remainder of this paper. Let $J=[0,b]$ with $b>0$ be a finite interval of the real line $\mathbb{R}$ and $C:=C(J,E)$ be the Banach space of all continuous functions $v$ from $J$ into $E$ with the supremum norm

$$\|\omega\|_\infty:=\sup_{t\in J}\|\omega(t)\|.$$

The notation $L^1(J,E)$ denotes the Banach space of measurable functions $u : J\to E$ which are Bochner integrable normed by

$$\|u\|_{L^1}=\int_0^b\|\omega(s)\|ds,\ \ \text{for all}\ \omega\in L^1(J,E).$$

As usual, $AC(J)$ denote the space of absolutely continuous function from $J$ into $E$. We denote by $AC^n_e(J)$ the space defined by

$$AC^n_e(J):=\left\{\omega : J\to E :\ {}^{e}D^{n-1}\omega(t)\in AC(J),\ {}^{e}D=e^{-t}\frac{d}{dt}\right\},$$

where $n=[\alpha]+1$ and $[\alpha]$ is the integer part of $\alpha$. In particular, if $0<\alpha\leq 1$, then $n=1$ and $AC^1_e(J):=AC_e(J)$.

**Definition 1** *( [26]) The exponential type fractional integral of order $\alpha > 0$ of a function $h \in L^1(J,E)$ is defined by*

$$({}^eI_0^{\alpha}h)(t) := \frac{1}{\Gamma(\alpha)}\int_0^t \left(e^t - e^s\right)^{\alpha-1} h(s)e^s ds, \text{ for each } t \in J,$$

*where $\Gamma(.)$ is the (Euler's) Gamma function defined by*

$$\Gamma(\xi) = \int_0^{\infty} t^{\xi-1}e^{-t}dt, \quad \xi > 0.$$

**Definition 2** *( [26]) Let $\alpha > 0$ and $h \in AC_e^n(J)$. The Caputo exponential type fractional derivatives of order $\alpha$ is defined by*

$$({}_c^eD_0^{\alpha}h)(t) := \frac{1}{\Gamma(n-\alpha)}\int_0^t \left(e^t - e^s\right)^{n-\alpha-1}\left(e^{-s}\frac{d}{ds}\right)^n h(s)\frac{ds}{e^{-s}}, \quad \text{for each } t \in J,$$

*where $n = [\alpha]+1$ and $[\alpha]$ is the integer part of $\alpha$. In particular, if $\alpha = 0$, then*

$$\left({}_c^eD_{(\cdot)}^0 h\right)(t) := h(t),$$

**Property 1** *( [26]) If $\alpha, \beta > 0$, then*

$${}^eI_0^{\alpha}\left(e^t - 1\right)^{\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\left(e^t - 1\right)^{\alpha+\beta}, \quad \text{for a.e. } t \in J.$$

**Property 2** *( [26]) If $\alpha > 0$ and $1 \le p < \infty$, then for $h \in L^p(J)$ we have*

$${}_c^eD_0^{\alpha}({}^eI_0^{\alpha}h)(t) = h(t).$$

**Property 3** *[26] Let $\alpha \ge 0$ and $n = [\alpha]+1$ and $h \in AC_e^n(J)$. Then we have the following formulas*

$${}^eI_0^{\alpha}({}_c^eD_0^{\alpha}h)(t) = h(t) - \sum_{k=0}^{n-1}\frac{(e^t-1)^k}{k!}\,{}^eD^k h(0).$$

**Lemma 4** *Let $\alpha > 0$, and $h \in AC_e^n(J)$. Then the differential equation*

$${}_c^eD_0^{\alpha}h(t) = 0$$

*has a solutions*

$$h(t) = \eta_0 + \eta_1(e^s-1) + \eta_2(e^s-1)^2 + \ldots + \eta_{n-1}(e^s-1)^{n-1},$$

*where $\eta_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$, $n = [\alpha]+1$.*

**Lemma 5** *Let $\alpha > 0$, and $h \in AC_e^n(J)$. Then*

$${}^eI_a^{\alpha}\left({}_c^eD_a^{\alpha}h\right)(t) = h(t) + \eta_0 + \eta_1(e^s-1) + \eta_2(e^s-1)^2 + \ldots + \eta_{n-1}(e^s-1)^{n-1},$$

*for each $\eta_i \in \mathbb{R}$, $i = 0,1,2,\ldots,n-1$ and $n = [\alpha]+1$.*

**Definition 3** *( [9]) Let $X$ be a Banach space and let $\mathscr{M}_X$ be the family of bounded subsets of $X$. The Kuratowski measure of noncompactness is the map $\mu : \mathscr{M}_X \to [0,\infty)$ defined by*

$$\mu(M) = \inf\{\varepsilon > 0 : M \subseteq \bigcup_{j=1}^{n} M_j,\ diam(M_j) \leq \varepsilon\}, \quad here\ M \in \mathscr{M}_X,$$

*where $M \in \mathscr{M}_X$. The map $\mu$ satisfies the following properties :*

- $\mu(M) = 0 \iff \overline{M}$ is compact ( $M$ is relatively compact).
- $\mu$ is equal to zero on every one element-set.
- $\mu(\overline{M}) = \mu(M)$.
- $M_1 \subset M_2 \Rightarrow \mu(M_1) \leq \mu(M_2)$.
- $\mu(Conv(M)) = \mu(M)$.
- $\mu(M_1 + M_2) \leq \mu(M_1) + \mu(M_2)$.
- $\mu(\lambda M) = |\lambda|\ \mu(M), k \in \mathbb{R}$.

**Lemma 6** *( [18]) If $V \subset C(J,E)$ is a bounded and equicontinuous set, then*
*(i) the function $t \to \mu(V(t))$ is continuous on $J$, and*

$$\mu_c(V) = \sup_{a \leq t \leq b} \mu(V(t)).$$

*(ii)* $\mu\left(\int_a^b \omega(s)ds : \omega \in V\right) \leq \int_a^b \mu(V(s))ds,$
*where $\mu$ is the Kuratowski measure of noncompactness and*

$$V(s) = \{\omega(s) : \omega \in V\},\ s \in J.$$

In the sequel we will make use of the following fixed point theorems.

**Theorem 7** *(Darbo's fixed point theorem [17]). Let $X$ be a Banach space. and $B$ be a bounded, closed, convex and nonempty subset of $X$. Suppose a continuous mapping $\Lambda : B \to B$ is such that for all closed subsets $D$ of $B$,*

$$\alpha(\Lambda(D)) \leq k\alpha(D),$$

*where $0 \leq k < 1$. Then $\Lambda$ has a fixed point in $B$.*

**Theorem 8** *(Monch's fixed point theorem [21]). Let $D$ be a bounded, closed and convex subset of a Banach space $X$ such that $0 \in D$, and let $\Lambda$ be a continuous mapping of $D$ into itself. If the implication*

$$V = \overline{conv}\Lambda(V)\ or\ V = \Lambda(V) \cup \{0\} \Rightarrow \mu(V) = 0, \tag{3}$$

*holds for every subset $V$ of $D$, then $\Lambda$ has a fixed point.*

## 3. EXISTENCE RESULTS

Let us start by defining what we mean by a solution of the problem (1)–(2).

**Definition 4** *A function $\omega \in AC_e(J,E)$ is said to be a solution of the problem (1)–(2) is $\omega$ satisfied equation (1) on $J$ and conditions (2).*

For the existence of solutions for the problem $(1)-(2)$, we need the following auxiliary lemmas :

**Lemma 9** *Let $0 < \rho \leq 1$ and $\xi : J \to E$ be a continuous function. Then the linear fractional boundary value problem*

$$ {}_c^eD_0^\rho \omega(t) = \xi(t), \text{ for each, } t \in J,\ 0 < \rho \leq 1, \tag{4}$$

$$ c_1\omega(0) + c_2\omega(b) = \delta, \tag{5}$$

*where $c_1$, $c_2$, are real constants with $c_1 + c_2 \neq 0$, and $\delta \in E$ has a unique solution given by*

$$\begin{aligned} \omega(t) &= \frac{1}{\Gamma(\rho)}\int_0^t (e^t - e^s)^{\rho-1} e^s \xi(s)ds \\ &\quad - \frac{1}{(c_1+c_2)}\left[\frac{c_2}{\Gamma(\rho)}\int_0^b (e^b - e^s)^{\rho-1} e^s \xi(s)ds - \delta\right]. \end{aligned}$$

**Proof.** By integrating the formula (4), we get

$$ \omega(t) = \omega_0 + \frac{1}{\Gamma(\rho)}\int_0^t (e^t - e^s)^{\rho-1} e^s \xi(s)ds. \tag{6}$$

By (6), we get $c_1\omega(0) = c_1\omega_0$, and

$$ c_2\omega(b) = c_2\omega_0 + \frac{c_2}{\Gamma(\rho)}\int_0^b (e^b - e^s)^{\rho-1} e^s \xi(s)ds.$$

Then by condition (5), we deduce

$$ \omega_0 = -\frac{1}{(c_1+c_2)}\left[\frac{c_2}{\Gamma(\rho)}\int_0^b (e^b - e^s)^{\rho-1} e^s \xi(s)ds - \delta\right].$$

Replacing in (6), we get

$$ \omega(t) = \frac{1}{\Gamma(\rho)}\int_0^t (e^t - e^s)^{\rho-1} e^s \xi(s)ds - \frac{1}{(c_1+c_2)}\left[\frac{c_2}{\Gamma(\rho)}\int_0^b (e^b - e^s)^{\rho-1} e^s \xi(s)ds - \delta\right]. \quad \square$$

**Lemma 10** *Let a function $f(t,u,v) : J \times E \times E \to E$ be continuous. Then the problem (1)–(2) is equivalent to the problem :*

$$ \omega(t) = \Psi + {}^eI_0^\rho \vartheta(t) \tag{7}$$

*where $\vartheta \in C(J,E)$ satisfies the functional equation :*

$$ \Psi = \frac{1}{(c_1+c_2)}\left[\delta - \frac{c_2}{\Gamma(\rho)}\int_0^b (e^b - e^s)^{\rho-1} e^s \vartheta(s)ds\right]$$

*and*

$$ \vartheta(t) = f\Big(t, \Psi + {}^eI_0^\rho \vartheta(t), \vartheta(t)\Big).$$

**Proof.** Let $\omega$ be a solution of (7). Then $\omega(0) = \Psi$ and

$$ \omega(b) = \Psi + \frac{1}{\Gamma(\rho)}\int_0^b (e^b - e^s)^{\rho-1} e^s \vartheta(s)ds.$$

So,

$$ c_1\omega(0) + c_2\omega(b) = c_1\Psi + \left[c_2\Psi + \frac{c_2}{\Gamma(\rho)}\int_0^b (e^b - e^s)^{\rho-1} e^s \vartheta(s)ds\right]$$

$$
\begin{aligned}
&= (c_1+c_2)\Psi+\frac{c_2}{\Gamma(\rho)}\int_0^b (e^b-e^s)^{\rho-1}e^s\vartheta(s)ds \\
&= \frac{(c_1+c_2)}{(c_1+c_2)}\left[\delta-\frac{c_2}{\Gamma(\rho)}\int_0^b (e^b-e^s)^{\rho-1}e^s\vartheta(s)ds\right] \\
&\quad +\frac{c_2}{\Gamma(\rho)}\int_0^b (e^b-e^s)^{\rho-1}e^s\vartheta(s)ds \\
&= \delta.
\end{aligned}
$$

Then

$$c_1\omega(0)+c_2\omega(b)=\delta.$$

On the other hand, we have

$$
\begin{aligned}
{}^e_cD_0^\rho\omega(t) &= {}^e_cD_0^\rho\Big(\Psi+{}^eI_0^\rho\vartheta(t)\Big)=\vartheta(t) \\
&= f\Big(t,y(t),{}^e_cD_0^\rho\omega(t)\Big).
\end{aligned}
$$

Thus, $\omega$ is a solution of the problem (1)–(2). □

The following hypotheses will be used in the sequel :

$(H_1)$ The function $t\to f(t,u,v)$ is measurable on $J$ for each $u,v\in E$, and the functions $u\to f(t,u,v)$ and $v\to f(t,u,v)$ are continuous on $E$ for a.e. $t\in J$.

$(H_2)$ There exist constants $\ell_1>0$ and $0<\ell_2<1$ such that

$$\|f(t,u,v)-f(t,\bar{u},\bar{v})\|\le\ell_1\|u-\bar{u}\|+\ell_2\|v-\bar{v}\|,\ \text{ for any } u,v,\bar{u},\bar{v}\in E,\ t\in J.$$

**Remark 1** *( [6]) Conditions $(H_2)$ is equivalent to the inequality*

$$\mu\big(f(t,B_1,B_2)\big)\le\ell_1\mu(B_1)+\ell_2\mu(B_2),$$

*for any bounded sets $B_1,B_2\subseteq E$ and for each $t\in J$.*

Now, we are in a position to state and prove our existence result for the problem (1)-(2) based on Darbo's fixed point theorem.
Set

$$\phi=\frac{\ell_1}{1-\ell_2},\ \Theta=\frac{(e^b-1)^\rho}{\Gamma(\rho+1)}\left(1+\frac{|c_2|}{|c_1+c_2|}\right)\quad\text{and}\quad\overline{f}=\sup_{t\in J}\|f(t,0,0)\|.$$

**Theorem 11** *Assume $(H_1)$ and $(H_2)$ holds. If*

$$\phi\Theta<1, \tag{8}$$

*then BVP (1)-(2) has at least one solution on $J$.*

**Proof.** Transform the problem (1)-(2) into a fixed point problem. Define the operator $\Lambda:C(J,E)\to C(J,E)$ by

$$
\begin{aligned}
\Lambda(\omega)(t) &= \frac{\delta}{c_1+c_2}+\frac{1}{\Gamma(\rho)}\int_0^t (e^t-e^s)^{\rho-1}e^s\vartheta(s)ds \\
&\quad -\frac{c_2}{(c_1+c_2)\Gamma(\rho)}\int_0^b (e^b-e^s)^{\rho-1}e^s\vartheta(s)ds,
\end{aligned}
\tag{9}
$$

where $\vartheta \in C(J,E)$ such that

$$\vartheta(t) = f(t, \omega(t), \vartheta(t)).$$

**Step 1 :** $\Lambda$ *is continuous.*
Let $\{u_n\}$ be a sequence such that $u_n \to u$ in $C(J,E)$. Then, for each $t \in J$ :

$$\begin{aligned} \|\Lambda(u_n)(t) - \Lambda(u)(t)\| &\leq \frac{1}{\Gamma(\rho)} \int_0^t (e^t - e^s)^{\rho-1} e^s \|\vartheta_n(s) - \vartheta(s)\| ds \\ &+ \frac{|c_2|}{|c_1 + c_2|\Gamma(\rho)} \int_0^b (e^b - e^s)^{\rho-1} e^s \|\vartheta_n(s) - \vartheta(s)\| ds, \end{aligned}$$

where $\vartheta_n, \vartheta \in C(J,E)$ such that

$$\vartheta_n(t) = f(t, u_n(t), \vartheta_n(t)), \quad \text{and} \quad \vartheta(t) = f\big(t, u(t), \vartheta(t)\big).$$

Since $u_n \to u$ as $n \to \infty$ and $f$ is continuous, then by Lebesgue dominated convergence theorem, we have $\|\vartheta_n(t) - \vartheta(t)\| \to 0$ as $n \to \infty$, which leads to

$$\|\Lambda(u_n)(t) - \Lambda(u)(t)\|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

Consequently, $\Lambda$ is continuous. Before the next step, we consider the ball $B_R = \{u \in C(J,E) : \|u\|_\infty \leq R\}$, such that

$$R \geq \left[\frac{|\delta|}{|c_1 + c_2|} + \frac{\overline{f}}{1 - \ell_2}\Theta\right]\left[1 - \Theta\phi\right]^{-1}, \tag{10}$$

**Step 2 :** $\Lambda(B_R) \subset B_R$. Let $u \in B_R$ we show that $\Lambda u \in B_R$. We have, for each $t \in J$

$$\begin{aligned} \|\Lambda u(t)\| &\leq \frac{|\delta|}{|c_1 + c_2|} + \frac{1}{\Gamma(\rho)} \int_0^t (e^t - e^s)^{\rho-1} e^s \|\vartheta(s)\| ds \\ &+ \frac{|c_2|}{|c_1 + c_2|\Gamma(\rho)} \int_0^b (e^b - e^s)^{\rho-1} e^s \|\vartheta(s)\| ds. \end{aligned} \tag{11}$$

By condition $(H_2)$, for each $t \in J$, we have that

$$\begin{aligned} \|\vartheta(t)\| &= \|f(t, u(t), \vartheta(t))\| \\ &\leq \|f(t, u(t), \vartheta(t)) - f(t,0,0)\| + \|f(t,0,0)\| \\ &\leq \ell_1 \|u(t)| + \ell_2 \|\vartheta(t)\| + \|f(t,0,0)\| \\ &\leq \ell_1 R + \ell_2 \|\vartheta(t)\| + \overline{f}. \end{aligned}$$

Then

$$\begin{aligned} \|\vartheta(t)\| &\leq \frac{\ell_1 R + \overline{f}}{1 - \ell_2} \\ &= \phi R + \frac{\overline{f}}{1 - \ell_2} := \widetilde{M}. \end{aligned}$$

Thus, (10) and (11) implies that

$$\begin{aligned}
\|\Lambda u(t)\| &\leq \frac{|\delta|}{|c_1+c_2|}+\left(\phi R+\frac{\overline{f}}{1-\ell_2}\right)\frac{(e^b-1)^\rho}{\Gamma(\rho+1)} \\
&+ \left(\phi R+\frac{\overline{f}}{1-\ell_2}\right)\frac{|c_2|(e^b-1)^\rho}{|c_1+c_2|\Gamma(\rho+1)} \\
&\leq \frac{|\delta|}{|c_1+c_2|}+\frac{\phi(e^b-1)^\rho}{\Gamma(\rho+1)}\left[1+\frac{|c_2|}{|c_1+c_2|}\right]R \\
&+ \left(\frac{\overline{f}}{1-\ell_2}\right)\frac{(e^b-1)^\rho}{\Gamma(\rho+1)}\left[1+\frac{|c_2|}{|c_1+c_2|}\right] \\
&\leq \frac{|\delta|}{|c_1+c_2|}+\phi\Theta R+\left(\frac{\overline{f}}{1-\ell_2}\right)\Theta \\
&\leq R.
\end{aligned}$$

Then $\|\Lambda u\|_\infty \leq R$. Thus $\Lambda(B_R)\subset B_R$.

**Step 3 :** $\Lambda(B_R)$ *is bounded and equicontinuous*.
Let $\tau_1,\tau_2\in J,\ \tau_1<\tau_2$, and let $u\in B_R$. Then

$$\begin{aligned}
\|\Lambda(u)(\tau_2)-\Lambda(u)(\tau_1)\| &= \Bigg\|\frac{1}{\Gamma(\rho)}\int_0^{\tau_1}\left[(e^{\tau_2}-e^s)^{\rho-1}-(e^{\tau_1}-e^s)^{\rho-1}\right]e^s\vartheta(s)ds \\
&\quad +\frac{1}{\Gamma(\rho)}\int_{\tau_1}^{\tau_2}(e^{\tau_2}-e^s)^{\rho-1}e^s\vartheta(s)ds\Bigg\| \\
&\leq \frac{1}{\Gamma(\rho)}\int_0^{\tau_1}\left|(e^{\tau_2}-e^s)^{\rho-1}-(e^{\tau_1}-e^s)^{\rho-1}\right|e^s\|\vartheta(s)\|ds \\
&\quad +\frac{1}{\Gamma(\rho)}\int_{\tau_1}^{\tau_2}\left|(e^{\tau_2}-e^s)^{\rho-1}\right|e^s\|\vartheta(s)\|ds \\
&\leq \frac{\widetilde{M}}{\Gamma(\rho+1)}\left[(e^{\tau_1}-1)^\rho-(e^{\tau_2}-1)^\rho+2(e^{\tau_2}-e^{\tau_1})^\rho\right].
\end{aligned}$$

As $\tau_1\to\tau_2$, the right-hand side of the above inequality tends to zero.

**Step 4 :** *The operator* $\Lambda: B_R\to B_R$ *is a contraction*.
Let $V\subset B_R$ and $t\in J$, then we have

$$\begin{aligned}
\mu(\Lambda(V)(t)) &= \mu\Big(\{(\Lambda y)(t), y\in V\}\Big) \\
&\leq \frac{1}{\Gamma(\rho)}\Big\{\int_0^t(e^t-e^s)^{\rho-1}e^s\mu(\vartheta(s))ds, y\in V\Big\}.
\end{aligned}$$

Then for each $s \in J$, the Remark 1 implies that

$$\begin{aligned} \mu\Big(\{\vartheta(s), y \in V\}\Big) &= \mu\Big(\{f(s, y(s), \vartheta(s)), y \in V\}\Big) \\ &\leq \ell_1 \mu\Big(\{y(s), y \in V\}\Big) + \ell_2 \mu\Big(\{\vartheta(s), y \in V\}\Big). \end{aligned}$$

Thus,

$$\begin{aligned} \mu\Big(\{\vartheta(s), y \in V\}\Big) &\leq \frac{\ell_1}{1-\ell_2}\mu\Big(\{y(s), y \in V\}\Big) \\ &\leq \phi\ \mu\Big(\{y(s), y \in V\}\Big). \end{aligned}$$

Then

$$\begin{aligned} \mu\Big(\Lambda(V)(t)\Big) &\leq \frac{\phi}{\Gamma(\rho)}\left\{\int_0^t (e^t - e^s)^{\rho-1} e^s \{\mu(y(s))\} ds,\ y \in V\right\} \\ &\leq \frac{\phi \mu_c(V)}{\Gamma(\rho)} \int_0^t (e^t - e^s)^{\rho-1} e^s ds \\ &\leq \frac{\phi (e^b - 1)^{\rho}}{\Gamma(\rho+1)} \mu_c(V). \end{aligned}$$

Therefore

$$\mu_c(\Lambda V) \quad \leq \quad \frac{\phi (e^b - 1)^{\rho}}{\Gamma(\rho+1)} \mu_c(V).$$

So, by (8), the operator $\Lambda$ is a contraction. As a consequence of Theorem 7, we deduce that $\Lambda$ has a fixed point, which is solution to the problem $(1)-(2)$. This completes the proof. □

The following hypotheses will be used in the sequel :

$(H_4)$ There exists a continuous function $p : J \to [0, \infty)$ such that

$$\|f(t, u, v)\| \leq \frac{p(t)}{1 + \|u\| + \|v\|}, \quad \text{for a.e. } t \in J \text{ and } u, v \in E.$$

$(H_5)$ For each bounded set $B \subset E$ and for each $t \in J$, we have

$$\mu\Big(f\left(t, B,\ {}^{e}_{c}D^{\alpha}_{s_k} B\right)\Big) \leq p(t)\,\mu(B).$$

where ${}^{e}_{c}D^{\alpha}_{s_k} B = \left\{{}^{e}_{c}D^{\alpha}_{s_k} u : u \in B\right\}.$

Set

$$p^* := \sup_{t \in J} p(t), \quad \rho = \frac{(e^b - 1)^{\rho}}{\Gamma(\rho+1)}.$$

The second existence is based on the concept of measure of noncompactness and Mönch's fixed point theorem.

**Theorem 12** *Assume that $(H_1)$, $(H_4)$ and $(H_5)$ are satisfied. If*

$$p^* \rho < 1, \tag{12}$$

*then the BVP (1)-(2) has at least one solution on $J$.*

**Proof.** Consider the operator $\Lambda$ defined in (9). We shall show that $\Lambda$ satisfies the assumption of Mönch's fixed point theorem. The proof will be given in several steps.

**Step 1 :** $\Lambda$ is continuous.
Let $\{u_n\}$ be a sequence such that $u_n \to v$ in $C(J,E)$. Then, for each $t \in J$, we have,

$$\begin{aligned}\|\Lambda(u_n)(t)-\Lambda(u)(t)\| &\leq \frac{1}{\Gamma(\rho)}\int_0^t (e^t-e^s)^{\rho-1}e^s\|\vartheta_n(s)-\vartheta(s)\|ds \\ &\quad +\frac{|c_2|}{|c_1+c_2|\Gamma(\rho)}\int_0^b (e^b-e^s)^{\rho-1}e^s\|\vartheta_n(s)-\vartheta(s)\|ds,\end{aligned}$$

where $\vartheta_n, \vartheta \in C(J,E)$ such that

$$\vartheta_n(t)=f(t,u_n(t),\vartheta_n(t)) \quad \text{and} \quad \vartheta(t)=f\big(t,u(t),\vartheta(t)\big).$$

Since $u_n \to u$ as $n\to\infty$ and $f$ is continuous, then by Lebesgue dominated convergence theorem, we have $\|\vartheta_n(t)-\vartheta(t)\| \to 0$ as $n\to\infty$, which leads to

$$\|\Lambda(u_n)(t)-\Lambda(u)(t)\|_\infty \to 0 \quad \text{as} \quad n\to\infty.$$

Therefore $\Lambda$ is continuous. Before the next step, we consider the ball $B_{R_2}=\{y\in C(J,E): \|u\|_\infty \leq R_2\}$ where

$$R_2 \geq \frac{|\delta|}{|c_1+c_2|}+p^*\Theta.$$

**Step 2 :** Prove that for any $u \in B_{R_2}$, $\Lambda$ maps $B_{R_2}$ into itself.

$$\begin{aligned}\|\Lambda u(t)\| &\leq \frac{|\delta|}{|c_1+c_2|}+\frac{1}{\Gamma(\rho)}\int_0^t (e^t-e^s)^{\rho-1}e^s\|\vartheta(s)\|ds \\ &\quad +\frac{|c_2|}{|c_1+c_2|\,\Gamma(\rho)}\int_0^b (e^b-e^s)^{\rho-1}e^s\|\vartheta(s)\|ds \\ &\leq \frac{|\delta|}{|c_1+c_2|}+\frac{1}{\Gamma(\rho)}\int_0^t (e^t-e^s)^{\rho-1}e^s\|p(s)\|ds \\ &\quad +\frac{|c_2|}{|c_1+c_2|\,\Gamma(\rho)}\int_0^b (e^b-e^s)^{\rho-1}e^s\|p(s)\|ds \\ &\leq \frac{|\delta|}{|c_1+c_2|}+\frac{p^*}{\Gamma(\rho)}\int_0^t (e^t-e^s)^{\rho-1}e^s ds+\frac{|c_2|p^*}{|c_1+c_2|\,\Gamma(\rho)}\int_0^b (e^b-e^s)^{\rho-1}e^s ds \\ &\leq \frac{|\delta|}{|c_1+c_2|}+\frac{p^*(e^b-1)^\rho}{\Gamma(\rho+1)}+\frac{p^*\,|c_2|\,(e^b-1)^\rho}{|c_1+c_2|\,\Gamma(\rho+1)}\end{aligned}$$

$$\leq \frac{|\delta|}{|c_1+c_2|}+p^*\Theta.$$

Hence $\|\Lambda u\|_\infty \leq R_2$, for each $t\in J$. This implies that $\Lambda$ transforms the ball $B_{R_2}$ into itself.

**Step 3 :** $\Lambda(B_{R_2})$ is bounded.
Since $\Lambda(B_{R_2})\subseteq B_{R_2}$ and $B_{R_2}$ is bounded, then $\Lambda(B_{R_2})$ is bounded.

**Step 4 :** $\Lambda(B_{R_2})$ is equicontinuous.

Let $\tau_1,\tau_2\in J,\ \tau_1<\tau_2$, and let $u\in B_R$. Then

$$\begin{aligned}
\|\Lambda(u)(\tau_2)-\Lambda(u)(\tau_1)\| &= \left\|\frac{1}{\Gamma(\rho)}\int_0^{\tau_1}\left[(e^{\tau_2}-e^s)^{\rho-1}-(e^{\tau_1}-e^s)^{\rho-1}\right]e^s\vartheta(s)ds\right.\\
&\quad \left.+\frac{1}{\Gamma(\rho)}\int_{\tau_1}^{\tau_2}(e^{\tau_2}-e^s)^{\rho-1}e^s\vartheta(s)ds\right\|\\
&\leq \frac{1}{\Gamma(\rho)}\int_0^{\tau_1}\left|(e^{\tau_2}-e^s)^{\rho-1}-(e^{\tau_1}-e^s)^{\rho-1}\right|e^s\|\vartheta(s)\|ds\\
&\quad +\frac{1}{\Gamma(\rho)}\int_{\tau_1}^{\tau_2}\left|(e^{\tau_2}-e^s)^{\rho-1}\right|e^s\|\vartheta(s)\|ds\\
&\leq \frac{p^*}{\Gamma(\rho+1)}\left[(e^{\tau_1}-1)^\rho-(e^{\tau_2}-1)^\rho+2(e^{\tau_2}-e^{\tau_1})^\rho\right].
\end{aligned}$$

As $\tau_1\to\tau_2$, the right-hand side of the above inequality tends to zero. Hence, $\Lambda(B_{R_2})$ is equicontinuous.

**Step 5 :** The implication (3) holds.
Now let $V$ be a subset of $B_{R_2}$ such that $V\subset\overline{conv}(\Lambda(V)\cup\{0\})$. $V$ is bounded and equicontinuous and therefore the function $t\to v(t)=\mu(V(t))$ is continuous on $J$.
By using the Lemma 6 and Properties of measure of noncompactness $\mu$, we have, for each $t\in J$,

$$\begin{aligned}
v(t) &\leq \mu(\Lambda(V)(t)\cup\{0\})\\
&\leq \mu(\Lambda(V)(t))\\
&\leq \frac{1}{\Gamma(\rho)}\int_0^t(e^t-e^s)^{\rho-1}e^sp(s)\mu\left(V(s)\right)ds\\
&\leq \frac{p^*}{\Gamma(\rho)}\int_0^t(e^t-e^s)^{\rho-1}e^sv(s)ds\\
&\leq p^*\,\rho\,\|v\|_\infty.
\end{aligned}$$

Therefore,

$$\|v\|_\infty \leq p^*\,\rho\,\|v\|_\infty.$$

From (12), we get $v(t) = 0$ for each $t \in J$, and then $V(t)$ is relatively compact in $E$. In view of the Ascoli-Arzelà theorem, $V$ is relatively compact in $B_{R_2}$. Applying now Theorem 8 we conclude that $\Lambda$ has a fixed point $u \in B_{R_2}$, which is solution to the problem $(1)-(2)$. This completes the proof. $\square$

## 4. ULAM-HYERS-RASIAS STABILITY

In this section, we are concerned with Ulam-Hyers-Rasias (**U-H-R**) stability. So, we adopt the definition in Rus [22] to our problem (1)-(2).

**Definition 5** *Equation* $(1)$ *is U-H-R stable with respect to* $\varphi \in C(J, \mathbb{R}_+)$ *if there exists a real number* $c_f > 0$ *such that for each* $\varepsilon > 0$ *and for each solution* $\varpi \in AC_e(J,E)$ *of the inequality*

$$\|{}_c^eD_0^\rho \varpi(t) - f(t, \varpi(t),\ {}_c^eD_0^\rho \varpi(t))\| \leq \varepsilon\varphi(t),\ t \in J, \tag{13}$$

*there exists a solution* $\omega \in AC_e(J,E)$ *of problem (1)-(2) with*

$$\|\varpi(t) - \omega(t)\|_E \leq c_f \varepsilon\varphi(t),\ t \in J.$$

**Remark 2** *A function* $\varpi \in AC_e(J,E)$ *is a solution of the inequality* $(13)$ *if and only if there exists a function* $g \in C(J,E)$ *(which depend on* $\varpi$*) such that*

*(i)* $\|g(t)\| \leq \varepsilon\varphi(t)$, *for* $t \in J$.

*(ii)* ${}_c^eD_0^\rho \varpi(t) = f(t, \varpi(t),\ {}_c^eD_0^\rho \varpi(t)) + g(t)$, *for* $t \in J$.

**Theorem 13** *Assume* $(H_1)$, $(H_2)$, *(8) and*

$(H_6)$ *There exists an increasing function* $\varphi \in C(J, \mathbb{R}_+)$ *and there exists* $\lambda_\varphi > 0$ *such that for each* $t \in J$, *we have*

$$^eI_0^\rho \varphi(t) \leq \lambda_\varphi \varphi(t)$$

*are satisfied. Then the problem (1)-(2) is U-H-R stable with respect to* $\varphi$.

**Proof.** Let $\varpi$ be a solution of the following inequality

$$\|{}_c^eD_0^\rho \varpi(t) - f(t, \varpi(t),\ {}_c^eD_0^\rho \varpi(t))\| \leq \varepsilon\varphi(t),\ t \in J. \tag{14}$$

Let us denote by $\omega$ the unique solution of the problem

$$ {}_c^eD_0^\rho \omega(t) = f(t, \omega(t),\ {}_c^eD_0^\rho \omega(t)), \text{ for each } t \in J,\ 0 < \rho \leq 1,$$

$$\omega(0) = \varpi(0),\ \omega(b) = \varpi(b).$$

By using Lemma 10, we have

$$\omega(t) \quad = \quad \Psi_\omega + \frac{1}{\Gamma(\rho)}\int_0^t (e^t - e^s)^{\rho-1} e^s \vartheta_\omega(s)ds,$$

where $\vartheta_\omega \in C(J,E)$ such that

$$\vartheta_\omega(t) = f(t, \Psi_\omega + {}^eI_0^\rho \vartheta_\omega(t), \vartheta_\omega(t))$$

and

$$\Psi_\omega = \frac{1}{(c_1+c_2)}\left[\delta - \frac{c_2}{\Gamma(\rho)}\int_0^b (e^b - e^s)^{\rho-1} e^s \vartheta_\omega(s)ds\right].$$

By integration of the formula (14), we obtain

$$\left\|\varpi(t) - \Psi_\varpi - \frac{1}{\Gamma(\rho)}\int_0^t (e^t - e^s)^{\rho-1} e^s \vartheta_\varpi(s)ds\right\| \quad \leq \quad \frac{\varepsilon}{\Gamma(\rho)}\int_0^b (e^b - e^s)^{\rho-1} e^s \varphi(s)ds$$

$$\leq \quad \varepsilon\lambda_\varphi \varphi(t). \tag{15}$$

We have, for each $t \in J$

$$\begin{aligned}
\|\varpi(t)-\omega(t)\| &= \left\|\varpi(t)-\Psi_\omega-\frac{1}{\Gamma(\rho)}\int_0^t (e^t-e^s)^{\rho-1}e^s\vartheta_\omega(s)ds\right\| \\
&= \left\|\varpi(t)-\Psi_\varpi-\frac{1}{\Gamma(\rho)}\int_0^t (e^t-e^s)^{\rho-1}e^s\vartheta_\varpi(s)ds\right. \\
&\quad \left. +\frac{1}{\Gamma(\rho)}\int_0^t (e^t-e^s)^{\rho-1}e^s\left(\vartheta_\varpi(s)-\vartheta_\omega(s)\right)ds\right\| \\
&\leq \left\|\varpi(t)-\Psi_\varpi-\frac{1}{\Gamma(\rho)}\int_0^t (e^t-e^s)^{\rho-1}e^s\vartheta_\varpi(s)ds\right\| \\
&\quad +\frac{1}{\Gamma(\rho)}\int_0^t (e^t-e^s)^{\rho-1}e^s\|\vartheta_\varpi(s)-\vartheta_\omega(s)\|ds.
\end{aligned}$$

Indeed, by $(H_2)$, we have, for each $t \in J$

$$\begin{aligned}
\|\vartheta_\varpi(t)-\vartheta_\omega(t)\| &= \|f(t,\varpi(t),\vartheta_\varpi(t))-f(t,\omega(t),\vartheta_\omega(t))\| \\
&\leq \ell_1\|\varpi(t)-\omega(t)\|+\ell_2\|\vartheta_\varpi(t)-\vartheta_\omega(t)\|.
\end{aligned}$$

Then

$$\|\vartheta_\varpi(t)-\vartheta_\omega(t)\| \leq \phi\|\varpi(t)-\omega(t)\|. \tag{16}$$

Using (15) and (16), we obtain

$$\|\varpi(t)-\omega(t)\| \leq \varepsilon\lambda_\varphi\varphi(t)+\frac{\phi}{\Gamma(\rho)}\int_0^t (e^t-e^s)^{\rho-1}e^s\|\varpi(s)-\omega(s)\|ds.$$

Then

$$\begin{aligned}
\|\varpi(t)-\omega(t)\| &\leq \varepsilon\lambda_\varphi\varphi(t)+\frac{\phi\parallel\varpi-\omega\parallel_E}{\Gamma(\rho)}\int_0^t (e^t-e^s)^{\rho-1}e^s ds \\
&\leq \varepsilon\lambda_\varphi\varphi(t)+\frac{\phi\parallel\varpi-\omega\parallel_E(e^b-1)^\rho}{\Gamma(\rho+1)}.
\end{aligned}$$

So

$$\parallel\varpi-\omega\parallel_E \quad \leq \quad \varepsilon\lambda_\varphi\varphi(t)+\frac{\phi\parallel\varpi-\omega\parallel_E(e^b-1)^\rho}{\Gamma(\rho+1)}.$$

Thus,

$$||\varpi-\omega||_E\left[1-\frac{\phi(e^b-1)^\rho}{\Gamma(\rho+1)}\right] \quad \leq \quad \varepsilon\lambda_\varphi\varphi(t).$$

From the condition (8), it follows that

$$||\varpi-\omega||_E \quad \leq \quad \varepsilon\lambda_\varphi\varphi(t)\left[1-\frac{\phi(e^b-1)^\rho}{\Gamma(\rho+1)}\right]^{-1}.$$

Then, for each $t \in J$

$$\|\varpi(t)-\omega(t)\|_E \quad \leq \quad \lambda_\varphi \varepsilon \varphi(t)\left[1-\frac{\phi(e^b-1)^\rho}{\Gamma(\rho+1)}\right]^{-1} := c_f \varepsilon \varphi(t).$$

Therefore, the problem (1)-(2) is U-H-R stable with respect to $\varphi$. This completes the proof. □

**Remark 3** *Our results for the boundary value problem (1)-(2) remain true for the following cases :*

- *Initial value problem :* $c_1 = 1, c_2 = 0$ *and* $\delta$ *arbitrary.*
- *Terminal value problem :* $c_1 = 0, c_2 = 1$ *and* $\delta$ *arbitrary.*
- *Anti-periodic problem :* $c_1 = c_2 \neq 0$ *and* $\delta = 0$.

*However, our results are not applicable for the periodic problem, i.e. for* $c_1 = 1,\ c_2 = -1$, *and* $\delta = 0$.

## 5. AN EXAMPLE

In this section, we will give an example to illustrate our main results. Let

$$E = l^1 = \left\{\omega = (\omega_1, \omega_2, \ldots, \omega_n, \ldots) : \sum_{n=1}^{\infty} |\omega_n| < \infty\right\}$$

be the Banach space with the norm

$$\|\omega\|_E = \sum_{n=1}^{\infty} |\omega_n|.$$

Consider the following boundary value problem for the nonlinear implicit fractional differential equation :

$$ {}^e_cD^{\frac{1}{2}}_0 \omega_n(t) = \frac{(3+|\omega_n(t)|+|{}^cD^{\frac{1}{2}}\omega_n(t)|)}{3e^{t+200}(1+|\omega_n(t)|+|{}^cD^{\frac{1}{2}}\omega_n(t)|)}, \text{ for each, } t \in [0,1], \tag{17}$$

$$\omega_n(0)+\omega_n(1) = 1. \tag{18}$$

where $J = [0,1],\ b = 1,\ c_1 = c_2 = \delta = 1,\ \omega = (\omega_1, \omega_2, \ldots, \omega_n, \ldots),\ f = (f_1, f_2, \ldots, f_n, \ldots)$, ${}^e_cD^{\frac{1}{2}}_0 \omega = \left({}^e_cD^{\frac{1}{2}}_0 \omega_1,\ {}^e_cD^{\frac{1}{2}}_0 \omega_2, \ldots,\ {}^e_cD^{\frac{1}{2}}_0 \omega_n, \ldots\right)$ and

$$f(t,u,v) = \frac{(3+||u||+||v||)}{3e^{t+200}(1+||u||+||v||)},\ \ t \in [0,1],\ u,v \in E.$$

For any $u, v, \bar{u}, \bar{v} \in E$ and $t \in [0,1]$, we can show that

$$||f(t,u,v)-f(t,\bar{u},\bar{v})|| \leq \frac{2}{3e^{200}}\left(\|u-\bar{u}\|_E + \|v-\bar{v}\|_E\right).$$

Thus, for $\ell_1 = \ell_2 = \dfrac{2}{3e^{200}}$, we have

$$\phi\Theta = \frac{(e^b-1)^\rho}{\Gamma(\rho+1)}\left(1+\frac{|\mu|}{|\lambda+\mu|}\right)\frac{\ell_1}{1-\ell_2} = \sqrt{\frac{e-1}{\pi}} \times \frac{6}{3e^{200}-2} \approx 2.047 \times 10^{-87} < 1,$$

Hence, from Theorem 11. The boundary value problem (17)-(18) has at least one solution on $[0,1]$. On the other hand, with the choice of $\psi(t) = e^t - 1$. We find that

$$^{e}I_0^{\frac{1}{2}}\psi(t) = \frac{4}{3\sqrt{\pi}}\sqrt{e^t-1}\left(e^t-1\right) \leq \frac{4\sqrt{e}}{3\sqrt{\pi}}\,\psi(t).$$

Thus, $(H_6)$ is satisfied with $\lambda_\psi = \dfrac{4\sqrt{e}}{3\sqrt{\pi}}$. Therefore, From Theorem 13, the BVP (17)-(18) is U-H-R stable with respect to $\psi$.

## 6. REFERENCES

[1] S. Abbas, M. Benchohra, J.R. Graef and J. Henderson, *Implicit Fractional Differential and Integral Equations : Existence and Stability,* De Gruyter, Berlin, 2018.

[2] S. Abbas, M. Benchohra, J.E. Lazreg and G. N'Guérékata, Hilfer and Hadamard functional random fractional differential inclusions, *Cubo* **19** (2017), 17-38.

[3] S. Abbas, M. Benchohra, J.E. Lazreg and Y. Zhou, A survey on Hadamard and Hilfer fractional differential equations : analysis and stability, *Chaos Solitons Fractals* **102** (2017), 47-71.

[4] S. Abbas, M. Benchohra and G. M. N'Guérékata, *Advanced Fractional Differential and Integral Equations*, Nova Science Publishers, New York, 2015.

[5] S. Abbas, M. Benchohra and G. M. N'Guérékata, *Topics in Fractional Differential Equations*, Springer, New York, 2012.

[6] J. Appell, Implicit Functions, Nonlinear Integral Equations, and the Measure of Noncompactness of the superposition Operator. *J. Math. Anal. Appl.* **83** (1981), 251-263.

[7] D. Baleanu, K. Diethelm, E. Scalas and J. J. Trujillo, *Fractional Calculus Models and Numerical Methods*, World Scientific, Singapore, 2012.

[8] D. Baleanu, Z. B. Guvenc, J. A. Tenreiro Machado, (eds.) : *New Trends in Nanotechnology and Fractional Calculus Applications*. Springer, Dordrecht, 2010.

[9] J. Banas and K. Goebel, *Measures of noncompactness in Banach spaces*. Marcel Dekker, New York, 1980.

[10] M. Benchohra and S. Bouriah, Existence and Stability Results for Nonlinear Boundary Value Problem for Implicit Differential Equations of Fractional Order, *Moroccan J. Pure Appl. Anal.(MJPAA).* **1** (1) (2015), 22-37.

[11] M. Benchohra, S. Bouriah and M.Darwish, Nonlinear boundary value problem for implicit differential equations of fractional order in Banach spaces, *Fixed Point Theory*. **18** (2) (2017), 457-470.

[12] M. Benchohra and J. E. Lazreg, Existence and Ulam stability for nonlinear implicit fractional differential equations with Hadamard derivative, *Stud. Univ. Babes-Bolyai Math.* **62** (2017), 27-38.

[13] M. Benchohra and J. E. Lazreg, On stability for nonlinear implicit fractional differential equations, *Matematiche (Catania)* **70** (2015), 49-61.

[14] M. Benchohra and J. E. Lazreg, Nonlinear fractional implicit differential equations. *Commun. Appl. Anal.* **17** (2013), 471-482.

[15] M. Benchohra and J. E. Lazreg, Existence and uniqueness results for nonlinear implicit fractional differential equations with boundary conditions. *Romanian J. Math. Comput. Sc.* **4** (1) (2014), 60-72.

[16] Z. Denton, J. D. Ramirez, Existence of minimal and maximal solutions to RL fractional integro-differential initial value problems. *Opuscula Math.* **37** (2017), no. 5, 705-724.

[17] K. Goebel, *Concise Course on Fixed Point Theorems*, Yokohama Publishers, Japan, 2002.

[18] D.J. Guo, V. Lakshmikantham, X. Liu, *Nonlinear Integral Equations in Abstract Spaces,* Kluwer Academic Publishers, Dordrecht, 1996.

[19] A.A. Kilbas, Hari M. Srivastava, and Juan J. Trujillo, *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[20] A. I. N. Malti, M. Benchohra, J. R. Graef, J.E. Lazreg, Impulsive boundary problems for nonlinear implicit Caputo-exponential type fractional differential equations, *Electron. J. Qual. Theory Differ. Equ.,* **2020** (78) (2020), 1-17.

[21] H. Mönch, Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces, *Nonlinear Anal.* **4** (1980), 985-999.

[22] I. A. Rus, Ulam stabilities of ordinary differential equations in a Banach space, *Carpathian J. Math.,* **26** (2010), 103-107.

[23] A. Salim, M. Benchohra , J. E. Lazreg and G. N'Guérékata, Boundary Value Problem for Nonlinear Implicit Generalized Hilfer-Type Fractional Differential Equations with Impulses. *Abstract and Applied Analysis.* **2021** (2021), 17pp.

[24] A. Salim, M. Benchohra, J. E. Lazreg, J. J. Nieto and Y. Zhou, Nonlocal Initial Value Problem for Hybrid Generalized Hilfer-type Fractional Implicit Differential Equations. *Nonauton. Dyn. Syst.* **8** (2021), 87-100.

[25] A. Salim, M. Benchohra, J. R. Graef and J. E. Lazreg, Boundary value problem for fractional generalised Hilfer-type fractional derivative with non-instantaneous impulses. *Fractal Fract.* **5** (2021), 1-21.

[26] J. Tariboon and S.K. Ntouyas, Initial and boundary value problems via exponential fractional calculus *Int. Electron. J. Pure Appl. Math.* **12** (2018), no. 1, 1-19.

[27] Y. Zhou, J.-R. Wang, L. Zhang, *Basic Theory of Fractional Differential Equations*. Second edition. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2017.