VARIABLE BESOV-TYPE SPACES

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ABSTRACT

In this paper we introduce Besov-type spaces with variable smoothness and integrability. We show that these spaces are characterized by the φ-transforms in appropriate sequence spaces and we obtain atomic decompositions for these spaces. Moreover the Sobolev embeddings for these function spaces are obtained.

1. INTRODUCTION

Besov spaces of variable smoothness and integrability, $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$, initially appeared in the paper of Almeida and Hästö [1]. Several basic properties were established, such as the Fourier analytical characterization and Sobolev embeddings. When p, q, α are constants they coincide with the usual function spaces $B^{\alpha}_{p,q}$. Later, [7] characterized these spaces by local means and established the atomic characterization. Afterwards, Kempka and Vybíral [16] characterized these spaces by the ball means of differences and also by local means, see [14] for the duality of $B^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ spaces.

Variable Besov-type spaces have been introduced in [9] and [10], where their basic properties are given, such as the Sobolev type embeddings and that under some conditions these spaces are just the variable Besov spaces. For constant exponents, these spaces unify and generalize many classical function spaces including Besov spaces, Besov-Morrey spaces (see, for example, [25, Corollary 3.3]). Independently, D. Yang, C. Zhuo and W. Yuan, [23] studied these function spaces where several properties are obtained such as atomic decomposition and the boundedness of trace operator. Also, Tyulenev [20], [21] has studied a new function spaces of variable smoothness. Triebel-Lizorkin spaces with variable smoothness and integrability $F^{\alpha(\cdot)}_{p(\cdot),q(\cdot)}$ were introduced in [4]. They proved a discretization by the so called φ-transform. Also atomic and molecular decompositions of these function spaces are obtained and used it to derive trace results. Subsequently, Vybiral [22] established Sobolev-Jawerth embeddings of these spaces. In [24], Triebel-Lizorkin type spaces of variable smoothness and integrability were introduced and studied. Their function spaces generalize classical Triebel-Lizorkin-type spaces and Triebel-Lizorkin spaces with variable smoothness and integrability.

The motivation to study such function spaces comes from applications to other fields of applied mathematics, such that fluid dynamics and image processing, see, for example, [17].

The main aim of this paper is to present another Besov-type spaces with variable smoothness and integrability which covers Besov-type spaces with fixed exponents. We establish their φ-transform characterization in the sense of Frazier and Jawerth. We also characterize these spaces by smooth atoms and give some basic properties and Sobolev-type embeddings.

2. PRELIMINARIES

As usual, we denote by $\mathbb{R}^n$ the n-dimensional real Euclidean space, $\mathbb{N}$ the collection of all natural numbers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The letter $\mathbb{Z}$ stands for the set of all integer numbers. The expression $f \lesssim g$ means that $f \leq c\, g$ for some independent constant c (and non-negative functions f and g), and $f \approx g$ means $f \lesssim g \lesssim f$. As usual for any $x \in \mathbb{R}$, $[x]$ stands for the largest integer smaller than or equal to x.

By supp f we denote the support of the function f, i.e., the closure of its non-zero set. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for the (Lebesgue) measure of E and χ_E denotes its characteristic function.

The Hardy-Littlewood maximal operator $\mathscr{M}$ is defined on $L^1_{\text{loc}}(\mathbb{R}^n)$ by

$$\mathscr{M} f(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy$$

and

$$M_B(f) := \frac{1}{|B|} \int_B |f(y)|\, dy.$$

The symbol $\mathscr{S}(\mathbb{R}^n)$ is used in place of the set of all Schwartz functions on $\mathbb{R}^n$. We denote by $\mathscr{S}'(\mathbb{R}^n)$ the dual space of all tempered distributions on $\mathbb{R}^n$. The Fourier transform of a tempered distribution f is denoted by $\mathscr{F} f$ while its inverse transform is denoted by $\mathscr{F}^{-1} f$.

For $v \in \mathbb{Z}$ and $m = (m_1, ..., m_n) \in \mathbb{Z}^n$, let $Q_{v,m}$ be the dyadic cube in $\mathbb{R}^n$, $Q_{v,m} = \{(x_1, ..., x_n) : m_i \leq 2^v x_i < m_i + 1, i = 1, 2, ..., n\}$. For the collection of all such cubes we use

$$\mathscr{Q} := \{Q_{v,m} : v \in \mathbb{Z}, m \in \mathbb{Z}^n\}.$$

For each cube Q, we denote its center by c_Q, its lower left-corner by $x_{Q_{v,m}} = 2^{-v} m$ of $Q = Q_{v,m}$ and its side length by $l(Q)$. For $r > 0$, we denote by rQ the cube concentric with Q having the side length $rl(Q)$. Furthermore, we put $v_Q = -\log_2 l(Q)$ and $v_Q^+ = \max(v_Q, 0)$.

For $v \in \mathbb{Z}$, $\varphi \in \mathscr{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, we set $\widetilde{\varphi}(x) := \overline{\varphi(-x)}$, $\varphi_v(x) := 2^{vn} \varphi(2^v x)$, and

$$\varphi_{v,m}(x) := 2^{v\frac{n}{2}} \varphi(2^v x - m) = |Q_{v,m}|^{\frac{1}{2}} \varphi_v(x - x_{Q_{v,m}}) \quad \text{if} \quad Q = Q_{v,m}.$$

By c we denote generic positive constants, which may have different values at different occurrences. Although the exact values of the constants are usually irrelevant for our purposes, sometimes we emphasize their dependence on certain parameters (e.g. $c(p)$ means that c depends on p, etc.). Further notation will be properly introduced whenever needed.

The variable exponents that we consider are always measurable functions p on $\mathbb{R}^n$ with range in $[c, \infty[$ for some $c > 0$. We denote the set of such functions by $\mathscr{P}_0$. The subset of variable exponents with range $[1, \infty[$ is denoted by $\mathscr{P}$. We use the standard notation $p^- := \underset{x \in \mathbb{R}^n}{\text{ess-inf}}\, p(x)$ and $p^+ := \underset{x \in \mathbb{R}^n}{\text{ess-sup}}\, p(x)$.

The variable exponent modular is defined by

$$\rho_{p(\cdot)}(f) := \int_{\mathbb{R}^n} \rho_{p(x)}(|f(x)|) dx,$$

where $\rho_p(t) = t^p$. The variable exponent Lebesgue space $L^{p(\cdot)}$ consists of measurable functions f on $\mathbb{R}^n$ such that $\rho_{p(\cdot)}(\lambda f) < \infty$ for some $\lambda > 0$. We define the Luxemburg (quasi)-norm on this space by the formula

$$\|f\|_{p(\cdot)} := \inf\Big\{\lambda > 0 : \rho_{p(\cdot)}\Big(\frac{f}{\lambda}\Big) \leq 1\Big\}.$$

A useful property is that $\|f\|_{p(\cdot)} \leq 1$ if and only if $\rho_{p(\cdot)}(f) \leq 1$, see, for example, [5], Lemma 3.2.4.

Let $p, q \in \mathscr{P}_0$. The mixed Lebesgue-sequence space $\ell^{q(\cdot)}(L^{p(\cdot)})$ is defined on sequences of $L^{p(\cdot)}$-functions by the semi-modular

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) := \sum_v \inf\Big\{\lambda_v > 0 : \rho_{p(\cdot)}\Big(\frac{f_v}{\lambda_v^{\frac{1}{q(\cdot)}}}\Big) \leq 1\Big\}.$$

The (quasi)-norm is defined from this as usual :

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} := \inf\Big\{\mu > 0 : \rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}\Big(\frac{1}{\mu}(f_v)_v\Big) \leq 1\Big\}. \tag{1}$$

If $q^+ < \infty$, then we can replace (1) by the simpler expression

$$\rho_{\ell^{q(\cdot)}(L^{p(\cdot)})}((f_v)_v) := \sum_v \big\||f_v|^{q(\cdot)}\big\|_{\frac{p(\cdot)}{q(\cdot)}}.$$

Furthermore, if p and q are constants, then $\ell^{q(\cdot)}(L^{p(\cdot)}) = \ell^q(L^p)$. The case $p := \infty$ can be included by replacing the last semi-modular by

$$\rho_{\ell^{q(\cdot)}(L^\infty)}((f_v)_v) := \sum_v \big\| |f_v|^{q(\cdot)} \big\|_\infty.$$

It is known, cf. [1, Theorem 3.6] and [15, Theorem 1], that $\ell^{q(\cdot)}(L^{p(\cdot)})$ is a norm if $q(\cdot) \geq 1$ is constant almost everywhere (a.e.) on $\mathbb{R}^n$ and $p(\cdot) \geq 1$, or if $\frac{1}{p(x)} + \frac{1}{q(x)} \leq 1$ a.e. on $\mathbb{R}^n$, or if $1 \leq q(x) \leq p(x) \leq \infty$ a.e. on $\mathbb{R}^n$.

We say that $g : \mathbb{R}^n \to \mathbb{R}$ is *locally* log-*Hölder continuous*, abbreviated $g \in C^{\log}_{\mathrm{loc}}$, if there exists $c_{\log}(g) > 0$ such that

$$|g(x) - g(y)| \leq \frac{c_{\log}(g)}{\log(e + \frac{1}{|x-y|})} \tag{2}$$

for all $x, y \in \mathbb{R}^n$. We say that g satisfies the log-*Hölder decay condition*, if there exists $g_\infty \in \mathbb{R}$ and a constant $c_{\log} > 0$ such that

$$|g(x) - g_\infty| \leq \frac{c_{\log}}{\log(e + |x|)}$$

for all $x \in \mathbb{R}^n$. We say that g is *globally*-log-*Hölder continuous*, abbreviated $g \in C^{\log}$, if it is locally log-Hölder continuous and satisfies the log-Hölder decay condition. The constants $c_{\log}(g)$ and $c_{\log}$ are called the *locally* log-*Hölder constant* and the log-*Hölder decay constant*, respectively. We note that all functions $g \in C^{\log}_{\mathrm{loc}}$ always belong to L^∞.

We define the following class of variable exponents

$$\mathscr{P}^{\log} := \Big\{p \in \mathscr{P} : \frac{1}{p} \in C^{\log}\Big\},$$

were introduced in [6, Section 2]. We define $\frac{1}{p_\infty} := \lim_{|x| \to \infty} \frac{1}{p(x)}$ and we use the convention $\frac{1}{\infty} = 0$. Note that although $\frac{1}{p}$ is bounded, the variable exponent p itself can be unbounded. It was shown in [5], Theorem 4.3.8 that $\mathscr{M} : L^{p(\cdot)} \to L^{p(\cdot)}$ is bounded if $p \in \mathscr{P}^{\log}$ and $p^- > 1$, see also [6], Theorem 1.2. Also if $p \in \mathscr{P}^{\log}$, then the convolution with a radially decreasing L^1-function is bounded on $L^{p(\cdot)}$:

$$\|\varphi * f\|_{p(\cdot)} \leq c\|\varphi\|_1 \|f\|_{p(\cdot)}.$$

We also refer to the papers [2] and [3], where various results on maximal function in variable Lebesgue spaces were obtained.

It is known that for $p \in \mathscr{P}^{\log}$ we have

$$\|\chi_B\|_{p(\cdot)}\|\chi_B\|_{p'(\cdot)} \approx |B|. \tag{3}$$

with constants only depending on the log-Hölder constant of p (see, for example, [5, Section 4.5]). Here p' denotes the conjugate exponent of p given by $\frac{1}{p(\cdot)} + \frac{1}{p'(\cdot)} = 1$.

Recall that $\eta_{v,m}(x) := 2^{nv}(1+2^v|x|)^{-m}$, for any $x \in \mathbb{R}^n$, $v \in \mathbb{N}_0$ and $m > 0$. Note that $\eta_{v,m} \in L^1$ when $m > n$ and that $\|\eta_{v,m}\|_1 = c_m$ is independent of v, where this type of function was introduced in [13] and [5].

3. MAIN RESULTS

We set

$$\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{\tau(\cdot)}_{p(\cdot)})} := \sup_{P\in\mathscr{Q}}\Big\|\Big(\frac{f_v}{|P|^{\tau(\cdot)}}\chi_P\Big)_{v\geq v_P^+}\Big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})},$$

where, $v_P = -\log_2 l(P)$ and $v_P^+ = \max(v_P, 0)$.

The following lemma is the $\ell^{q(\cdot)}(L^{\tau(\cdot)}_{p(\cdot)})$-version of Lemma 4.7 from Almeida and Hästö [1] (we use it, since the maximal operator is in general not bounded on $\ell^{q(\cdot)}(L^{p(\cdot)})$, see [1, Example 4.1]).

Lemma 1 *Let $\tau \in C^{\log}_{\text{loc}}$, $\tau^- > 0$, $p \in \mathscr{P}^{\log}$, $q \in \mathscr{P}^{\log}_0$ with $0 < q^- \leq q^+ < \infty$ and $\tau^+ < (\tau p)^-$. For any m large enough, there exists $c > 0$ such that*

$$\|(\eta_{v,m} * f_v)_v\|_{\ell^{q(\cdot)}(L^{\tau(\cdot)}_{p(\cdot)})} \leq c\|(f_v)_v\|_{\ell^{q(\cdot)}(L^{\tau(\cdot)}_{p(\cdot)})}$$

for any $(f_v)_v \in \ell^{q(\cdot)}(L^{\tau(\cdot)}_{p(\cdot)})$.

Let $\widetilde{L^{\tau(\cdot)}_{p(\cdot)}}$ be the collection of functions $f \in L^{p(\cdot)}_{\text{loc}}(\mathbb{R}^n)$ such that

$$\|f\|_{\widetilde{L^{\tau(\cdot)}_{p(\cdot)}}} := \sup\Big\|\frac{f\chi_P}{|P|^{\tau(\cdot)}}\Big\|_{p(\cdot)} < \infty, \quad p \in \mathscr{P}_0, \quad \tau : \mathbb{R}^n \to \mathbb{R}^+,$$

where the supremum is taken over all dyadic cubes P with $|P| \geq 1$. Notice that

$$\|f\|_{\widetilde{L^{\tau(\cdot)}_{p(\cdot)}}} \leq 1 \Leftrightarrow \sup_{P\in\mathscr{Q},|P|\geq 1}\Big\|\Big|\frac{f}{|P|^{\tau(\cdot)}}\Big|^{q(\cdot)}\chi_P\Big\|_{p(\cdot)/q(\cdot)} \leq 1. \tag{4}$$

Recall that $\theta_v = 2^{vn}\theta(2^v\cdot), v \in \mathbb{Z}$.

Lemma 2 *Let $v \in \mathbb{Z}$, $\tau \in C^{\log}_{\text{loc}}$, $\tau^- > 0$, $p \in \mathscr{P}^{\log}_0$ and $\theta, \omega \in \mathscr{S}(\mathbb{R}^n)$ with $\operatorname{supp}\mathscr{F}\omega \subset \overline{B(0,1)}$. For any $f \in \mathscr{S}'(\mathbb{R}^n)$ and any dyadic cube P with $|P| \geq 1$, we have*

$$\Big\|\frac{\theta_v * \omega_v * f}{|P|^{\tau(\cdot)}}\chi_P\Big\|_{p(\cdot)} \leq c\|\omega_v * f\|_{\widetilde{L^{\tau(\cdot)}_{p(\cdot)}}},$$

such that the right-hand side is finite, where $c > 0$ is independent of v and $l(P)$.

Lemma 3 *Let $\alpha,\tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- \geq 0$ and $p,q \in \mathscr{P}^{\log}_0$ with $0 < q^- \leq q^+ < \infty$. Let $(f_k)_{k\in\mathbb{N}_0}$ be a sequence of measurable functions on $\mathbb{R}^n$. For all $v \in \mathbb{N}_0$ and $x \in \mathbb{R}^n$, let*

$$g_v(x) := \sum_{k=0}^{\infty} 2^{-|k-v|\delta} f_k(x).$$

Then there exists a positive constant c, independent of $(f_k)_{k\in\mathbb{N}_0}$ such that

$$\big\|(g_v)_v\big\|_{\ell^{q(\cdot)}(L^{\tau(\cdot)}_{p(\cdot)})} \leq c\big\|(f_v)_v\big\|_{\ell^{q(\cdot)}(L^{\tau(\cdot)}_{p(\cdot)})}, \quad \delta > 0.$$

The proof of Lemma 3 can be obtained by the same arguments used in [10, Lemma 2.10] and [16, Lemma 8].

we present the Fourier analytical definition of Besov-type spaces of variable smoothness and integrability and we prove their basic properties in analogy to the Besov-type spaces with fixed exponents. Select a pair of Schwartz functions Φ and φ such that

$$\mathrm{supp}\mathscr{F}\Phi \subset \overline{B(0,2)} \quad \text{and} \quad |\mathscr{F}\Phi(\xi)| \geq c \quad \text{if} \quad |\xi| \leq \frac{5}{3} \tag{5}$$

and

$$\mathrm{supp}\mathscr{F}\varphi \subset \overline{B(0,2)}\backslash B(0,1/2) \quad \text{and} \quad |\mathscr{F}\varphi(\xi)| \geq c \quad \text{if} \quad \frac{3}{5} \leq |\xi| \leq \frac{5}{3}, \tag{6}$$

where $c > 0$. We put $\varphi_v := 2^{vn}\varphi(2^v\cdot), v \in \mathbb{N}$.

Definition 1 *Let $\alpha : \mathbb{R}^n \to \mathbb{R}$, $\tau : \mathbb{R}^n \to \mathbb{R}^+$ and $p,q \in \mathscr{P}_0$. Let Φ and φ satisfy* (5) *and* (6)*, respectively. The Besov-type space $\mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ is the collection of all $f \in \mathscr{S}'(\mathbb{R}^n)$ such that*

$$\big\|f\big\|_{\mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} := \sup_{P\in\mathscr{Q}} \Big\|\Big(\frac{2^{v\alpha(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}}\chi_P\Big)_{v\geq v_P^+}\Big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty, \tag{7}$$

where φ_0 is replaced by Φ.

Using the system $(\varphi_v)_{v\in\mathbb{N}_0}$ we can define the quasi-norm

$$\big\|f\big\|_{B^{\alpha,\tau}_{p,q}} := \sup_{P\in\mathscr{Q}} \frac{1}{|P|^{\tau}} \Big(\sum_{v=v_P^+}^{\infty} 2^{v\alpha q}\big\|(\varphi_v * f)\chi_P\big\|^q_p\Big)^{\frac{1}{q}}$$

for constants α and $p,q \in (0,\infty]$, with the usual modification if $q = \infty$. The Besov-type space $B^{\alpha,\tau}_{p,q}$ consist of all distributions $f \in \mathscr{S}'(\mathbb{R}^n)$ for which $\big\|f\big\|_{B^{\alpha,\tau}_{p,q}} < \infty$.

3.1. Some properties of variable Besov-type spaces

In the following theorem we have the possibility to define these spaces by replacing $v \geq v_P^+$ by $v \in \mathbb{N}_0$ in Definition 1, where the main arguments used in its proof rely on [9, Theorem 3.12], so we omit the details and when $\tau := 0$, was obtained by Sickel [18].

Theorem 4 *Let $\alpha,\tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- \geq 0$ and $p,q \in \mathscr{P}^{\log}_0$ with $p^+,q^+ < \infty$. If $(\tau p - 1)^+ < 0$ or $(\tau p - 1)^+ \leq 0$ and $q := \infty$, then*

$$\big\|f\big\|^{\blacktriangle}_{\mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} := \sup_{P\in\mathscr{Q}} \Big\|\Big(\frac{2^{\alpha(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}}\chi_P\Big)_{v\in\mathbb{N}_0}\Big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})},$$

is an equivalent quasi-norm in $\mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$.

For any $\gamma \in \mathbb{Z}$, we put

$$\|f\|^{*}_{\mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} := \sup_{P\in\mathcal{Q}} \Big\| \Big(\frac{2^{v\alpha(\cdot)}\varphi_v * f}{|P|^{\tau(\cdot)}} \chi_P \Big)_{v\geq v_P^+ - \gamma} \Big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} < \infty$$

where $\varphi_{-\gamma}$ is replaced by $\Phi_{-\gamma}$.

Lemma 5 *Let $\alpha, \tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- > 0$, $p, q \in \mathscr{P}^{\log}_0$ and $0 < q^+ < \infty$. The quasi-norms $\|f\|^{*}_{\mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$ and $\|f\|_{\mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$ are equivalent with equivalent constants depending on γ.*

Definition 2 *Let $p, q \in \mathscr{P}_0$, $\tau : \mathbb{R}^n \to \mathbb{R}^+$ and let $\alpha : \mathbb{R}^n \to \mathbb{R}$. Then we define*

$$\mathfrak{b}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} := \Big\{ \lambda = \{\lambda_{v,m}\}_{v\in\mathbb{N}_0, m\in\mathbb{Z}^n} \subset \mathbb{C} : \|\lambda\|_{\mathfrak{b}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} < \infty \Big\},$$

where

$$\|\lambda\|_{\mathfrak{b}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} := \sup_{P\in\mathcal{Q}} \Big\| \Big(\frac{\sum_{m\in\mathbb{Z}^n} 2^{v(\alpha(\cdot)+\frac{n}{2})} \lambda_{v,m}\chi_{v,m}}{|P|^{\tau(\cdot)}} \chi_P \Big)_{v\geq v_P^+} \Big\|_{\ell^{q(\cdot)}(L^{p(\cdot)})}.$$

If we replace dyadic cubes P by arbitrary balls B_J of $\mathbb{R}^n$ with $J \in \mathbb{Z}$, we then obtain equivalent quasi-norms, where the supremum is taken over all $J \in \mathbb{Z}$ and all balls B_J of $\mathbb{R}^n$.

Lemma 6 *Let $\alpha, \tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- \geq 0$, $p, q \in \mathscr{P}^{\log}_0$, $0 < q^+ < \infty$, $v \in \mathbb{N}_0, m \in \mathbb{Z}^n$, $x \in Q_{v,m}$ and $\lambda \in \mathfrak{b}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$. Then there exists $c > 0$ independent of x, v and m such that*

$$|\lambda_{v,m}| \leq c\, 2^{-v(\alpha(x)+\frac{n}{2})} |Q_{v,m}|^{\tau(x)} \|\lambda\|_{\mathfrak{b}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \|\chi_{v,m}\|^{-1}_{p(\cdot)}.$$

For a sequence $\lambda = \{\lambda_{v,m}\}_{v\in\mathbb{N}_0, m\in\mathbb{Z}^n} \subset \mathbb{C}, 0 < r \leq \infty$ and a fixed $d > 0$, set

$$\lambda^*_{v,m,r,d} := \Big(\sum_{h\in\mathbb{Z}^n} \frac{|\lambda_{v,h}|^r}{(1 + 2^v|2^{-v}h - 2^{-v}m|)^d} \Big)^{\frac{1}{r}}$$

and $\lambda^*_{r,d} := \{\lambda^*_{v,m,r,d}\}_{v\in\mathbb{N}_0, m\in\mathbb{Z}^n} \subset \mathbb{C}$.

Lemma 7 *Let $\alpha, \tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- > 0$, $p, q \in \mathscr{P}^{\log}_0$, $0 < q^+ < \infty$ and $0 < r < \frac{(\tau p)^-}{\tau^+}$. Then for d large enough*

$$\|\lambda^*_{r,d}\|_{\mathfrak{b}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} \approx \|\lambda\|_{\mathfrak{b}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}.$$

3.2. Embeddings

For the spaces $\mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$ introduced above we want to show some embedding theorems. We say a quasi-Banach space A_1 is continuously embedded in another quasi-Banach space A_2, $A_1 \hookrightarrow A_2$, if $A_1 \subset A_2$ and there is a $c > 0$ such that $\|f\|_{A_2} \leq c\|f\|_{A_1}$ for all $f \in A_1$. We begin with the following elementary embeddings.

Theorem 8 *Let $\alpha, \tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- > 0$ and $p, q, q_0, q_1 \in \mathscr{P}^{\log}_0$.*
(i) If $q_0 \leq q_1$, then

$$\mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q_0(\cdot)} \hookrightarrow \mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q_1(\cdot)}.$$

(ii) If $(\alpha_0 - \alpha_1)^- > 0$, then

$$\mathfrak{B}^{\alpha_0(\cdot),\tau(\cdot)}_{p(\cdot),q_0(\cdot)} \hookrightarrow \mathfrak{B}^{\alpha_1(\cdot),\tau(\cdot)}_{p(\cdot),q_1(\cdot)}.$$

The proof can be obtained by using the same method as in [1, Theorem 6.1]. We next consider embeddings of Sobolev-type. It is well-known that

$$B^{\alpha_0,\tau}_{p_0,q} \hookrightarrow B^{\alpha_1,\tau}_{p_1,q},$$

if $\alpha_0 - \frac{n}{p_0} = \alpha_1 - \frac{n}{p_1}$, where $0 < p_0 < p_1 \leq \infty, 0 \leq \tau < \infty$ and $0 < q \leq \infty$ (see e.g. [25, Corollary 2.2]). In the following theorem we generalize these embeddings to variable exponent case.

Theorem 9 *Let $\alpha_0, \alpha_1, \tau \in C^{\log}_{\mathrm{loc}}$, $\tau^- > 0$ and $p_0, p_1, q \in \mathscr{P}^{\log}_0$ with $q^+ < \infty$. If $\alpha_0(\cdot) > \alpha_1(\cdot)$ and $\alpha_0(\cdot) - \frac{n}{p_0(\cdot)} = \alpha_1(\cdot) - \frac{n}{p_1(\cdot)}$ with $\left(\frac{p_0}{p_1}\right)^+ < 1$, then*

$$\mathfrak{B}^{\alpha_0(\cdot),\tau(\cdot)}_{p_0(\cdot),q(\cdot)} \hookrightarrow \mathfrak{B}^{\alpha_1(\cdot),\tau(\cdot)}_{p_1(\cdot),q(\cdot)}.$$

Theorem 10 *Let $\alpha, \tau \in C^{\log}_{\mathrm{loc}}, \tau^- > 0$ and $p, q \in \mathscr{P}^{\log}_0$ with $q^+ < \infty$. Then*

$$\mathscr{S}(\mathbb{R}^n) \hookrightarrow \mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)} \hookrightarrow \mathscr{S}'(\mathbb{R}^n).$$

Now we establish some further embedding of the spaces $\mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$.

Theorem 11 *Let $\alpha, \tau \in C^{\log}_{\mathrm{loc}}, \tau^- > 0$ and $p, q \in \mathscr{P}^{\log}_0$ with $q^+ < \infty$. If $(p_2 - p_1)^+ \leq 0$, then*

$$\mathfrak{B}^{\alpha(\cdot)+n\tau(\cdot)+\frac{n}{p_2(\cdot)}-\frac{n}{p_1(\cdot)},0}_{p_2(\cdot),q(\cdot)} \hookrightarrow \mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p_1(\cdot),q(\cdot)}.$$

3.3. Atomic decomposition

The idea of atomic decompositions leads back to M. Frazier and B. Jawerth in their series of papers [11], [12]. The main goal of this section is to prove an atomic decomposition result for $\mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$. We define for $a > 0$, $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $f \in \mathscr{S}'(\mathbb{R}^n)$, the Peetre maximal function

$$\varphi_v^{*,a} 2^{v\alpha(\cdot)} f(x) := \sup_{y \in \mathbb{R}^n} \frac{2^{v\alpha(y)} |\varphi_v * f(y)|}{(1 + 2^v |x-y|)^a}, \qquad v \in \mathbb{N}_0.$$

where φ_0 is replaced by Φ. We now present a fundamental characterization of spaces under consideration.

$$\varphi_v^{*,a} 2^{v\alpha(\cdot)} f(x) := \sup_{y \in \mathbb{R}^n} \frac{2^{v\alpha(y)} |\varphi_v * f(y)|}{(1 + 2^v |x-y|)^a}, \qquad v \in \mathbb{N}_0.$$

where φ_0 is replaced by Φ. We now present a fundamental characterization of spaces under consideration.

Theorem 12 *Let $\tau, \alpha \in C^{\log}_{\mathrm{loc}}, \tau^- > 0$ and $p, q \in \mathscr{P}^{\log}_0$. Let m be as in Lemma 1, $a > \frac{m\tau^+}{(\tau p)^-}$ and Φ and φ satisfy* (5) *and* (6)*, respectively. Then*

$$\|f\|^{\blacktriangledown}_{\mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}} := \sup_{P \in \mathscr{Q}} \left\| \left(\frac{\varphi_v^{*,a} 2^{v\alpha(\cdot)} f}{|P|^{\tau(\cdot)}} \chi_P \right)_{v \geq v_P^+} \right\|_{\ell^{q(\cdot)}(L^{p(\cdot)})} \tag{8}$$

is an equivalent quasi-norm in $\mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$.

Atoms are the building blocks for the atomic decomposition.

Definition 3 *Let $K \in \mathbb{N}_0, L+1 \in \mathbb{N}_0$ and let $\gamma > 1$. A K-times continuous differentiable function $a \in C^K(\mathbb{R}^n)$ is called $[K,L]$-atom centered at $Q_{v,m}$, $v \in \mathbb{N}_0$ and $m \in \mathbb{Z}^n$, if*

$$\text{supp } a \subseteq \gamma Q_{v,m} \tag{9}$$

$$|\partial^\beta a(x)| \leq 2^{v(|\beta|+\frac{1}{2})}, \quad \text{for} \quad 0 \leq |\beta| \leq K, x \in \mathbb{R}^n \tag{10}$$

and if

$$\int_{\mathbb{R}^n} x^\beta a(x)dx = 0, \quad \text{for} \quad 0 \leq |\beta| \leq L \text{ and } v \geq 1. \tag{11}$$

If the atom a located at $Q_{v,m}$, that means if it fulfills (9), then we will denote it by $a_{v,m}$. For $v = 0$ or $L = -1$ there are no moment conditions (11) required.

Now we come to the atomic decomposition theorem.

Theorem 13 *[26] Let $\alpha, \tau \in C^{\log}_{\text{loc}}, \tau^- > 0$ and $p, q \in \mathscr{P}^{\log}_0$ with $0 < q^- \leq q^+ < \infty$. Let $0 < p^- \leq p^+ < \infty$ and let $K, L+1 \in \mathbb{N}_0$ such that*

$$K \geq ([\alpha^+ + n\tau^+] + 1)^+, \tag{12}$$

and

$$L \geq \max\Big(-1, [n(\frac{1}{\min(1, \frac{(\tau p)^-}{\tau^+})} - 1) - \alpha^-]\Big). \tag{13}$$

Then $f \in \mathscr{S}'(\mathbb{R}^n)$ belongs to $\mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$, if and only if it can be represented as

$$f = \sum_{v=0}^{\infty} \sum_{m \in \mathbb{Z}^n} \lambda_{v,m} \rho_{v,m}, \quad \text{converging in } \mathscr{S}'(\mathbb{R}^n), \tag{14}$$

where $\rho_{v,m}$ are $[K,L]$-atoms and $\lambda = \{\lambda_{v,m}\}_{v \in \mathbb{N}_0, m \in \mathbb{Z}^n} \in \mathfrak{b}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$. Moreover, $\inf \|\lambda\|_{\mathfrak{b}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}}$, where the infimum is taken over admissible representations (14)*, is an equivalent quasi-norm in $\mathfrak{B}^{\alpha(\cdot),\tau(\cdot)}_{p(\cdot),q(\cdot)}$.*

If p, q, τ, and α are constants, then the restriction (12), and their counterparts, in the atomic decomposition theorem are $K \geq ([\alpha + n\tau] + 1)^+$ and $L \geq \max\big(-1, [n(\frac{1}{\min(1,p)} - 1) - \alpha]\big)$, which are essentially the restrictions from the works of [8, Theorem 3.12].

4. CONCLUSIONS

In this work, we present some properties of variable Besov-type space. Moreover the Sobolev embeddings for these function spaces are obtained. We also establish the atomic decomposition of variable besov type space, and we obtain a convolution inequality for this space. The aim of our results is to study the boundedness of many operators in harmonic analysis, for example the pseudo-differential operator.

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