STABILITY OF AN ABSTRACT SYSTEM WITH INFINITE HISTORY

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ABSTRACT

This work is concerned with stabilization of an abstract linear dissipative integro-differential equation with infinite memory modeling linear viscoelasticity where the relaxation function satisfies $g'(t) \leq -\xi(t)g^p(t), \forall t \geq 0, 1 \leq p < \frac{3}{2}$. Our approach improves and has a better decay rate than the one in the literature.

1. INTRODUCTION

Let us denote by $\mathscr{H}$ a Hilbert space with inner product and related norm denoted by $\langle , \rangle$ and $\|.\|$ respectively. Let $A : \mathscr{D}(A) \longrightarrow \mathscr{H}$ and $B : \mathscr{D}(B) \longrightarrow \mathscr{H}$ be self-adjoint linear positive definite operators with domains $\mathscr{D}(A) \subset \mathscr{D}(B) \subset \mathscr{H}$ such that the embeddings are dense and compact.

We are interested in energy decay of the solution u to the following initial boundary value problem

$$u_{tt} + Au - \int_0^{+\infty} g(s)Bu(t-s)\,ds = 0, \qquad \forall t > 0, \tag{1}$$

with initial conditions

$$\begin{cases} u(-t) = u_0(t), & \forall t \in \mathbb{R}_+, \\ u_t(0) = u_1, & \end{cases} \tag{2}$$

where u_0 and u_1 are given history and initial data, g is a positive and nonincreasing function called the relaxation function.

By following the brilliant intuition of Dafermos [1, 2], we introduce the relative history of u defined as

$$\begin{cases} \eta^t(s) = u(t) - u(t-s), & \forall t, s \in \mathbb{R}_+, \\ \eta_0(s) = \eta^0(s) = u_0(0) - u_0(s), & \forall s \in \mathbb{R}_+. \end{cases}$$

Equation (1)-(2) can be rewritten as an abstract linear first-order system of the form

$$\begin{cases} \mathscr{U}_t + \mathscr{A}\mathscr{U}(t) = 0, & \forall t > 0, \\ \mathscr{U}(0) = \mathscr{U}_0, & \end{cases} \tag{3}$$

where $\mathscr{U}_0 = (u_0(0), u_1, \eta_0)^T \in \mathscr{H} = \mathscr{D}(A^{\frac{1}{2}}) \times H \times L^2_g(\mathbb{R}_+, \mathscr{D}(B^{\frac{1}{2}}))$, $\mathscr{U} = (u, u_t, \eta^t)^T$ and $L^2_g(\mathbb{R}_+, \mathscr{D}(B^{\frac{1}{2}}))$ is the weighted space with respect to the measure $g(s)ds$ defined by

$$L^2_g(\mathbb{R}_+, \mathscr{D}(B^{\frac{1}{2}})) = \left\{ z : \mathbb{R}_+ \longrightarrow \mathscr{D}(B^{\frac{1}{2}}),\ \int_0^{+\infty} g(s) \|B^{\frac{1}{2}} z(s)\|^2 \, ds < +\infty \right\}$$

endowed with the inner product

$$\langle z_1, z_2 \rangle_{L^2_g(\mathbb{R}_+, \mathscr{D}(B^{\frac{1}{2}}))} = \int_0^{+\infty} g(s) \langle B^{\frac{1}{2}} z_1(s), B^{\frac{1}{2}} z_2(s) \rangle \, ds.$$

The operator $\mathscr{A}$ is defined by

$$\mathscr{A}(v, w, z)^T = \left(-w, Av - g_0 Bv + \int_0^{+\infty} g(s) Bz(s) \, ds, \frac{\partial z}{\partial s} - w \right)^T,$$

where $g_0 = \int_0^{+\infty} g(s) \, ds$,

$$\mathscr{D}(\mathscr{A}) = \left\{ (v, w, z)^T \in \mathscr{H},\ v \in \mathscr{D}(A),\ w \in \mathscr{D}(A^{\frac{1}{2}})),\ z \in \mathscr{L}_g,\ \int_0^{+\infty} g(s) z(s) \, ds \in \mathscr{D}(B) \right\},$$

and $\mathscr{L}_g = \left\{ z \in L^2_g(\mathbb{R}_+, \mathscr{D}(B^{\frac{1}{2}})), \partial_s z \in L^2_g(\mathbb{R}_+, \mathscr{D}(B^{\frac{1}{2}})), z(0) = 0 \right\}$.

As shown in [10] for example, under the assumptions $(\mathbb{H}_1)$ and $(\mathbb{H}_2)$ below, the space $\mathscr{H}$ endowed with the inner product

$$\langle (v_1, w_1, z_1)^T, (v_2, w_2, z_2)^T \rangle_{\mathscr{H}} = \langle A^{\frac{1}{2}} v_1, A^{\frac{1}{2}} v_2 \rangle - g_0 \left\langle B^{\frac{1}{2}} v_1, B^{\frac{1}{2}} v_2 \right\rangle + \langle w_1, w_2 \rangle + \langle z_1, z_2 \rangle_{L^2_g(\mathbb{R}_+, \mathscr{D}(B^{\frac{1}{2}}))}$$

is a Hilbert space, $\mathscr{D}(\mathscr{A}) \subset \mathscr{H}$ with dense embedding, and $\mathscr{A}$ is the infinitesimal generator of a linear contraction $\mathscr{C}_0$ semigroup on $\mathscr{H}$. Therefore, the classical semigroup theory implies that (see [13]), for any $\mathscr{U}_0 \subset \mathscr{H}$, the system (3) has a unique weak solution

$$\mathscr{U} \in \mathscr{C}(\mathbb{R}_+, \mathscr{H}).$$

Moreover, if $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A})$, then the solution of (3) is classical; that is

$$\mathscr{U} \in C^1(\mathbb{R}_+, \mathscr{H}) \cap C(\mathbb{R}_+, \mathscr{D}(\mathscr{A})).$$

Problems related to (1)-(2) have been studied by many authors and several stability results have been established; see [3, 6, 11, 12]. The exponential and polynomial decay of the solutions of equation (1)-(2) have been studied in [5], where it was assumed that $(\mathbb{H}_1)$ holds and

- (A_1) There exists an increasing strictly convex function $G : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[)$ satisfying

$$G(0) = G'(0) = 0 \qquad and \qquad \lim_{t \longrightarrow +\infty} G'(t) = +\infty,$$

such that

$$\int_0^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} \, ds + \sup_{s \in \mathbb{R}_+} \frac{g(s)}{G^{-1}(-g'(s))} < +\infty.$$

The author established a general decay estimate given in term of the convex function G. His result generalizes the usual exponential and polynomial decay results found in the literature. He considered two cases corresponding to the following two conditions on A and B:

$$\exists a_2 > 0: \qquad \|A^{\frac{1}{2}}v\|^2 \le a_2\|B^{\frac{1}{2}}v\|^2, \quad \forall v \in \mathscr{D}(A^{\frac{1}{2}}). \tag{4}$$

Or

$$\exists a_2 > 0: \qquad \|A^{\frac{1}{2}}v\|^2 \le a_2\|A^{\frac{1}{2}}B^{\frac{1}{2}}v\|^2, \quad \forall v \in \mathscr{D}(A^{\frac{1}{2}}B^{\frac{1}{2}}). \tag{5}$$

The study of viscoelastic problem (1)-(2) in the particular case $A = B$ was considered by Guesmia and Messaoudi [4]. The authors considered (6) below with $p = 1$ and extended the decay result known for problems with finite history to those with infinite history. In addition, they improved, in some cases, some decay results obtained earlier in [5].

Very recently, the authors of [8] considered the condition

$$g'(t) \le -\xi(t)g^p(t), \qquad \forall t \in \mathbb{R}_+, \tag{6}$$

where ξ is a positive and nonincreasing function and $1 \le p < \frac{3}{2}$, with the objective of improving the decay rate for problems with finite memory.

Condition (6) gives a better description of the growth of g at infinity and allows to obtain a precise estimate of the energy that is more general than the "stronger " one (ξ constant and $p \in [1, \frac{3}{2}[$) used in the case of past history control [7, 9]. We also refer the reader to some recent researches under the condition (6) with finite history and viscolelastic term [8] for related results. The authors proved a general decay rate from which the exponential decay is only a special case. Moreover, the optimal polynomial decay is easily and directly obtained without restrictive conditions.

With the above motivations and inspired by the approach of [8], we intend to study the general decay result to problem (1)-(2) under suitable assumptions on the initial data and the relaxation function g. Our main contribution is an enhancement to the results of [4, 5] in a way that our result gives a better rate of decay in the polynomial case.

2. PRELIMINARIES AND MAIN RESULT

In this section, we shall present some necessary assumptions as well as the main result of our study. Let us assume that
($\mathbb{H}_1$) There exist positive constants a_0 and a_1 such that

$$a_1\|v\|^2 \le \|B^{\frac{1}{2}}v\|^2 \le a_0\|A^{\frac{1}{2}}v\|^2, \quad \forall v \in \mathscr{D}(A^{\frac{1}{2}}).$$

($\mathbb{H}_2$) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a differentiable nonincreasing function satisfying

$$0 < g_0 < \frac{1}{a_0}.$$

($\mathbb{H}_3$) There exists a nonincreasing differentiable function $\xi : \mathbb{R}_+ \to \mathbb{R}_+$ and $1 \le p < \frac{3}{2}$ satisfying (6).

Note that, for any regular solution u of the problem (1)-(2), it is straightforward to see that

$$E'(t) \quad = \quad \frac{1}{2}\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2\, ds, \quad \forall t \in \mathbb{R}_+, \tag{7}$$

where

$$E(t) = \frac{1}{2}\|\mathscr{U}(t)\|^2_{\mathscr{H}} = \frac{1}{2}\left(\|A^{\frac{1}{2}}u(t)\|^2 - g_0\|B^{\frac{1}{2}}u(t)\|^2 + \|u_t(t)\|^2 + \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2\, ds\right). \quad (8)$$

Theorem 1 *Assume that* $(\mathbb{H}_1)$, $(\mathbb{H}_2)$ *and* $(\mathbb{H}_3)$ *hold.*

1. *Let* $\mathscr{U}_0 \in H$ *and* $\mathscr{U}$ *be the solution of* (3).
If (4) *holds, and if,*

$$\exists m_0 > 0: \quad \|B^{\frac{1}{2}}u_0(s)\| \le m_0, \qquad \forall s > 0, \quad (9)$$

then there exists a positive constant C such that, for all $t \in \mathbb{R}_+$,

$$E(t) \le C(1+t)^{-\frac{1}{2p-2}}\xi^{-\frac{2p-1}{2p-2}}(t)\left[1 + \int_0^t (s+1)^{\frac{1}{2p-2}}\xi^{\frac{2p-1}{2p-2}}(s)h^{2p-1}(s)\, ds\right], \quad (10)$$

where $h(t) = \xi(t)\int_t^{+\infty} g(s)\, ds$.

Moreover, if

$$\int_0^{+\infty}(1+t)^{-\frac{1}{2p-2}}\xi^{-\frac{2p-1}{2p-2}}(t)\left[1 + \int_0^t (s+1)^{\frac{1}{2p-2}}\xi^{\frac{2p-1}{2p-2}}(s)h^{2p-1}(s)\, ds\right] dt < +\infty, \quad (11)$$

then, for all $t \in \mathbb{R}_+$,

$$E(t) \le C(1+t)^{-\frac{1}{p-1}}\xi^{-\frac{p}{p-1}}(t)\left[1 + \int_0^t (s+1)^{\frac{1}{p-1}}\xi^{\frac{p}{p-1}}(s)h^p(s)\, ds\right]. \quad (12)$$

2. *Let* $\mathscr{U}_O \in D(A) \times D(A^{\frac{1}{2}}) \times L^2_g(\mathbb{R}_+, D(A^{\frac{1}{2}}B^{\frac{1}{2}})$ *and* $\mathscr{U}$ *be the solution of* (3).
If (5) *holds, and if,*

$$\exists m_0 > 0: \quad \|A^{\frac{1}{2}}B^{\frac{1}{2}}u_0(s)\| \le m_0, \quad \forall s > 0, \quad (13)$$

then there exists a positive constant C such that, for all $t \in \mathbb{R}_+$,

$$E(t) \le C\left(\frac{E_2(0) + E^{2p-1}(0) + \int_0^t h^{2p-1}(s)\, ds}{\int_0^t \xi^{2p-1}(s)\, ds}\right)^{\frac{1}{2p-1}}, \quad (14)$$

where

$$E_2(t) = \frac{1}{2}\left(\|Au(t)\|^2 - g_0\|A^{\frac{1}{2}}B^{\frac{1}{2}}u(t)\|^2 + \|A^{\frac{1}{2}}u'(t)\|^2\right) + \frac{1}{2}\int_0^{+\infty} g(s)\|A^{\frac{1}{2}}B^{\frac{1}{2}}\eta^t(s)\|^2\, ds. \quad (15)$$

Moreover, if

$$\int_0^{+\infty}\left(\frac{E_2(0) + E^{2p-1}(0) + \int_0^t h^{2p-1}(s)\, ds}{\int_0^t \xi^{2p-1}(s)\, ds}\right)^{\frac{1}{2p-1}} < +\infty, \quad (16)$$

then for all $t \in \mathbb{R}_+$,

$$E(t) \le C\left(\frac{E_2(0) + E^p(0) + \int_0^t h^p(s)\, ds}{\int_0^t \xi^p(s)\, ds}\right)^{\frac{1}{p}}. \quad (17)$$

Now, we illustrate the energy decay rate given by Thoerem 1 through the following example :
Example: Let $g(t) = a(1+t)^{-q}, \quad q > 2$, where $a > 0$ is a constant so that $\int_0^{+\infty} g(t)\, dt < \frac{1}{a}$, then we have

$$g'(t) = -aq(1+t)^{-q-1} = -b(a(1+t)^{-q})^{\frac{q+1}{q}} = -bg^p(t), \quad p = \frac{q+1}{q},\, b > 0.$$

Therefore, for the Case (4), estimate (12) with $\xi(t) = b$ yields

$$E(t) \leq C(1+t)^{-\frac{1}{p-1}} \xi^{-\frac{p}{p-1}}(t) \left(1 + \int_0^t (s+1)^{\frac{1}{p-1}} \xi^{\frac{p}{p-1}}(s) h^p(s)\, ds\right).$$

Let us compute

$$\begin{aligned} h(t) = \xi(t) \int_t^{+\infty} g(s)\, ds &= b \int_t^{+\infty} a(1+s)^{-q}\, ds \\ &= \frac{ab}{q-1}(1+t)^{1-q}, \quad q = \frac{1}{p-1}. \end{aligned} \tag{18}$$

Observe that, for some positive constant C, it yields

$$\int_0^t (s+1)^{\frac{1}{p-1}} \xi^{\frac{p}{p-1}}(s) h^p(s)\, ds = b\frac{ab}{q-1} \int_0^t (1+s)^{p(1-q)+\frac{1}{p-1}}\, ds = C(1+t)^{p(1-q)+\frac{1}{p-1}+1} - C.$$

Then, it yields

$$\begin{aligned} E(t) &\leq C(1+t)^{-\frac{1}{p-1}} \xi^{-\frac{p}{p-1}}(t) \left(1 + C(1+t)^{p(1-q)+\frac{1}{p-1}+1}\right) \\ &= C(1+t)^{-\frac{1}{p-1}} + C(1+t)^{p(1-q)+1} \\ &= C(1+t)^{-q} + C(1+t)^{\frac{-q^2+q+1}{q}} \leq C(1+t)^{\frac{-q^2+q+1}{q}}. \end{aligned} \tag{19}$$

For the Case (5), estimate (17) gives

$$\begin{aligned} E(t) &\leq \left(\frac{E_2(0) + E^p(0) + \int_0^t h^p(s)\, ds}{\int_0^t \xi^p(s)\, ds}\right)^{\frac{q}{q+1}} \\ &\leq bt^{-\frac{q}{q+1}} \left(E_2(0) + E^p(0) + \int_0^t h^p(s)\, ds\right)^{\frac{q}{q+1}} \\ &\leq C\, t^{-\frac{q}{q+1}} \left(1 - (1+t)^{\frac{(1-q)(q+1)}{q}} + 1\right)^{\frac{q}{q+1}} \leq C\, t^{-\frac{q}{q+1}}. \end{aligned} \tag{20}$$

Let us compare our result with the one of [4, 5]. In this way, let us recall the approach of [5] with $B = A$, there exists a positive constant c_1 such that

$$E(t) \leq c_1(1+t)^{-c_2}, \qquad \forall t \in \mathbb{R}_+, \tag{21}$$

where c_2 is generated by the calculations and it is generally small.
Furthermore, the approach of [5] in polynomial case under (A_2) and with $G(t) = t^{\frac{1}{p}+1}$, for any $p \in \left]0, \frac{q-1}{2}\right[$ gives
If (4) holds,

$$E(t) \leq C(1+t)^{-p}, \quad \forall p \in \left]0, \frac{q-1}{2}\right[, \tag{22}$$

and if (5) holds,

$$E(t) \leq C(1+t)^{-\frac{p}{p+1}}, \quad \forall p \in \left]0, \frac{q-1}{2}\right[. \tag{23}$$

Now, since $\frac{q^2-q-1}{2} > \frac{q-1}{2}$ for $q > 2$. Then from (19), (21) and (22) , we conclude that our estimate (19) gives a better decay than (21) and (22).

For the case (5), we see that $\frac{q}{q+1} > \frac{p}{p+1}$, for any $p \in]0, \frac{q-1}{2}[$. Then estimate (20) has better decay than estimate (23) also for (19) under some hypothesis on dimension of space.

As a conclusion our approach improves and has a better decay rate than the one of [4, 5].

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