

POWER CONTAMINATION AND DOMINATION ON THE GRID

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ABSTRACT

The contamination game of a grid graph $G(n, m)$ is a dynamic variant of the domination [7], similar to the power domination [8]. This standard is introduced by Haynes, Hedetniemi and Henning in 2002 [6], which is initially defined as a basic domination for a set of vertices S in a graph G , and then a propagation of this domination in all vertices of G , while starting with S . On the other hand, the contamination phenomena in $G(n, m)$ is interpreted by an evolutionary automaton cellular, which aims to propagate viruses according to a given propagation rules. In this paper, we define a mathematical self-playing game called a contamination game based on the power domination, in which, we identify the minimum number of contaminant cells for $G(n, m)$, called the contamination number and denoted $\gamma_c(G(n, m))$.

1. INTRODUCTION

Electric power systems need to be monitored in real-time. One way to achieve this task is to place phase measurement units at selected locations in the system. The power system monitoring problem is a combinatorial optimization problem that consists of minimizing the number of measurement devices to be put in an electric power system. The power system monitoring problem has been formulated as a graph theory domination problem by Haynes, Hedetniemi, Hedetniemi, and Henning in [7]. This problem is of somehow different flavor than standard domination type problems, since putting a phase measurement unit into a vertex of a graph can have global effects. For instance, if an electric power system can be modeled by a path, then a single measurement unit suffices to monitor the system no matter how long is the path.

Let $G = (V, E)$ be a connected graph. For a vertex v in V , let $N(v)$ denote the open neighborhood of v , and for a subset S of V let $N(S) = (\cup_{v \in S} N(v)) \setminus S$. We denote with $M(S)$ the set monitored by S , defined algorithmically as follows [4] :

Algorithm 1 Construction of a monitored set $M(S)$

Input: Graph $G = (V, E)$ and $S \subset V$.

Output: $M(S)$, the monitored set by S .

- 1: Initiate $M(S) \leftarrow S \cup N(S)$;
 - 2: **While** there exists $v \in M(S)$ such that $N(v) \cap (V \setminus M(S)) = \{w\}$ **do**
 - 3: $M(S) \leftarrow M(S) \cup \{w\}$;
 - 4: **EndWhile** ;
 - 5: **Return** $M(S)$;
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The set S is called a power dominating set of G if $M(S) = V$ and the power domination number, denoted by $\gamma_\pi(G)$, is the minimum cardinality of a power dominating set. Various papers have addressed the power domination number, in which they essentially concentrate on its algorithmic point of view. This problem is proven to be NP-complete even when restricted to bipartite graphs, chordal graphs, planar graphs, circle graphs and split graphs [4]. In contrast, the problem can be solved in polynomial time for trees and interval graphs [4, 6]. Dorfling and Henning obtained closed formulas for the power domination numbers of grid graphs [5]. This result is in striking contrast with the fact that a determination of such formulas for the usual domination number of grid graphs is an open problem [7]. Now, a natural description of a grid is a cartesian product of two paths. However, there exist other graph products such as the strong, the direct, and the lexicographic product [7]. Hence, it is natural to ask whether the power domination number can also be determined for these products of paths.

In this paper we introduce a new variant of domination characterized as a virus-contamination in grid graph $G(n, m)$, which is defined in two steps :

- (1) Local domination for a few cells of $G(n, m)$.
- (2) Propagation on all cells of $G(n, m)$ according to a given initial contamination rules.

2. POWER CONTAMINATION ON THE GRID

Let $G(n, m) = (V, E)$ be a grid graph, and $S \subset V$. The set S is said to be a contaminating set if a full contamination of $G(n, m)$ can be achieved from S and the power contamination number $\gamma_c(G(n, m))$ is the minimum cardinality of a power contaminating set. In the following, we will illustrate the problem as a self-playing game, in order to deal with the problem of contamination in $G(n, m)$.

For a vertex v of $G(n, m)$, let $M(v)$ and $VN(v)$ denote, respectively, Moore neighborhood (Figure 1(a)) and Von Neumann neighborhood (Figure 1(b)) of v , extended to the cells at the edge of $G(n, m)$.

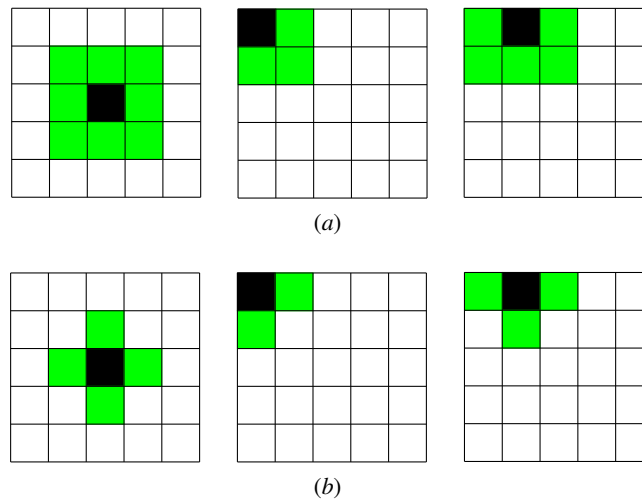


FIGURE 1 – Moore and Von Neumann neighborhoods of the black cell.

2.1. Contamination Rules in $G(n, m)$

The contamination game of $G(n, m)$ can be seen as a cellular automaton, or a model where each state leads automatically to the next state from predefined rules. This game takes place on $G(n, m)$, whose cells are considered by analogy as living cells, which can take two different states "sick" or "healthy". At each step, the state of any cell is determined by the state of its eight neighbors, in regards to a given initial contamination rules. The goal of this game is to find the minimum number of initial contaminated cells $\gamma_c(G(n, m))$, such that the entire grid is contaminated. This kind of contamination can be seen as an evolutionary cellular automaton, which models an epidemiological phenomenon, illustrating the propagation of viruses in living cells.

The space of states is a two-dimensional grid of sick or healthy living cells. The chosen transition rule depends on the number and position of the contaminated living neighboring cells that surround a cell, it corresponds to Moore neighborhood.

A cell v is contaminated by two sick cells v_1 and v_2 , if one of the following conditions is fulfilled :

- (1) $v_1, v_2 \in VN(v)$,
- (2) $v_1, v_2 \notin VN(v)$ and $M(v_1) \cap M(v_2) = \{v\}$.

The possible configurations which satisfies these conditions are given in Figure 2.

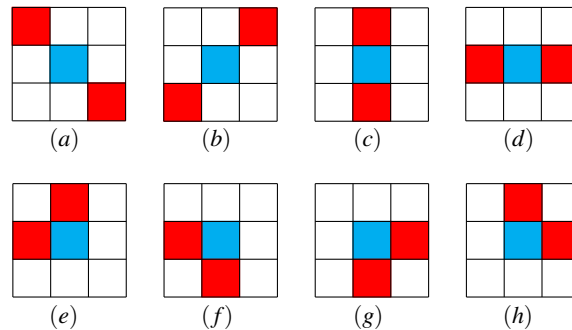


FIGURE 2 – The contamination rules of the blue cell.

The following algorithm illustrates the contamination and spread process which yield the contaminated set S , according to the contamination rules :

Algorithm 2 Construction of a contaminated set $C(S)$

Input: Graph $G = (V, E)$ and $S \subset V$.

Output: $C(S)$, the subset of vertices contaminated by S .

- 1: Initialize $C(S) \leftarrow S$;
 - 2: **While** $\exists v \in V \setminus C(S)$, such that the conditions (1) or (2) are satisfied **do**
 - 3: $C(S) \leftarrow C(S) \cup \{v\}$;
 - 4: **End While** ;
 - 5: **Return** $C(S)$;
-

3. CONTAMINATION ON STRONG PRODUCT OF TWO PATHS

A natural representation of a grid $G(n, m)$ is as the strong product of two paths $P_n \boxtimes P_m$, such that :

1. Each cell of $G(n, m)$ is represented by a vertex v in $P_n \boxtimes P_m$.
2. The neighboring between two cells in $G(n, m)$ is represented by an edge in $P_n \boxtimes P_m$.

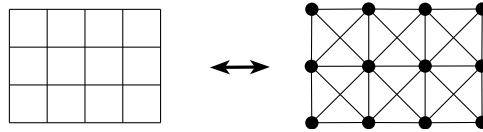


FIGURE 3 – $G(3,4)$ modeled as the strong product of paths

The number of neighboring of each cell in $G(n, m)$ represents the degree of the corresponding vertex in $P_n \boxtimes P_m$, as shown in Figure 3. This implies that the virus-contamination on $G(n, m)$ is equivalent as on $P_n \boxtimes P_m$.

Figure 4 represents an optimal contamination of $G(3,4)$. The red cells (equivalently the red vertices in $P_3 \boxtimes P_4$) represent the contaminated cells in step 0.

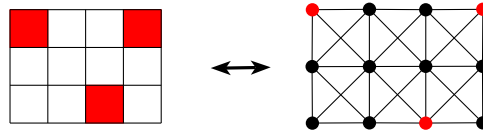


FIGURE 4 – $\gamma_c(G(3,4)) = 3$

The evolution of the total contamination of the grid $G(3,4)$ is shown in Figure 5.

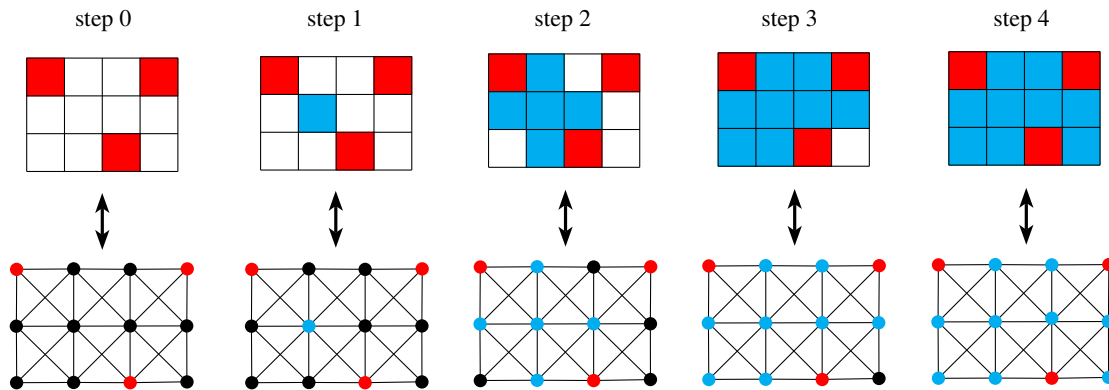


FIGURE 5 – The evolution of the total contamination of $G(3,4)$

4. MAIN RESULTS

Lemma 1 For any positive integer m , the contamination number of the path $P_m (= P_1 \boxtimes P_m)$ is :

$$\gamma_c(P_m) = 1 + \lfloor \frac{m}{2} \rfloor.$$

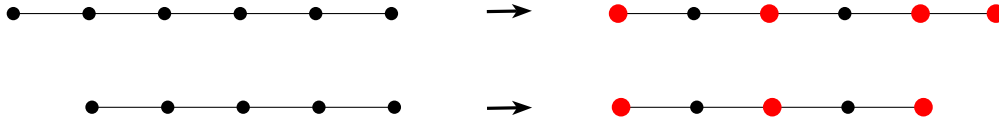


FIGURE 6 – Optimal contamination in P_6 and P_5

Theorem 2 Let n, m be two positive integers. Then we have

$$\gamma_{cv}(P_n \boxtimes P_m) \leq \begin{cases} \max \left\{ \lfloor \frac{n}{2} \rfloor, \lfloor \frac{m}{2} \rfloor \right\} + 1 & \text{if } m \text{ and } n \text{ have the same parity,} \\ \max \left\{ \lceil \frac{n}{2} \rceil, \lceil \frac{m}{2} \rceil \right\} + 1 & \text{else.} \end{cases}$$

As a consequence of the above theorem, we can give the following corollary.

Corollary 3 For any positive integer n , we have :

$$\gamma_{cv}(P_n \boxtimes P_n) \leq \lfloor \frac{n}{2} \rfloor + 1.$$

Our investigation therefore puts us in a position to conjecture the following result :

$$\gamma_{cv}(P_n \boxtimes P_m) = \begin{cases} \max \left\{ \lfloor \frac{n}{2} \rfloor, \lfloor \frac{m}{2} \rfloor \right\} + 1 & \text{if } m \text{ and } n \text{ have the same parity,} \\ \max \left\{ \lceil \frac{n}{2} \rceil, \lceil \frac{m}{2} \rceil \right\} + 1 & \text{else.} \end{cases}$$

5. CONCLUSION

In this work, we have introduced a new dynamic variant of domination, which has the same principle of unfolding as power domination. This type of domination can be interpreted as a biological phenomenon or an evolutionary social phenomenon, which is called a *contamination game* and takes place in the grid graph $G(n, m)$. We identified an upper bound for the minimum number of contaminant cells $\gamma_c(G(n, m))$ in $G(n, m)$ and conjectured that it gives the exact value.

6. REFERENCES

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