STRONG INCIDENCE COLOURING OF GRAPHS

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ABSTRACT

An incidence of a graph G is a pair (v,e) where v is a vertex of G and e is an edge of G incident with v. Two incidences (v,e) and (w,f) of G are adjacent whenever (i) $v=w$, or (ii) $e=f$, or (iii) $vw=e$ or f. An incidence p-colouring of G is a mapping from the set of incidences of G to the set of colours $\{1,\dots,p\}$ such that every two adjacent incidences receive distinct colours. Incidence colouring has been introduced by Brualdi and Quinn Massey in 1993 and, since then, studied by several authors.

In this paper, we introduce and study the strong version of incidence colouring, where incidences adjacent to a same incidence must also get distinct colours. We determine the exact value of – or upper bounds on – the strong incidence chromatic number of several classes of graphs, namely cycles, wheel graphs, trees, ladder graphs and subclasses of Halin graphs.

1. INTRODUCTION

All graphs considered in this paper are simple and loopless undirected graphs. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G, respectively, by $\Delta(G)$ the maximum degree of G, by $N(v)$ the set of vertices adjacent to the vertex v and by $\mathrm{dist}_G(u,v)$ the distance between vertices u and v in G.

An *incidence* of a graph G is a pair (v,e) where v is a vertex of G and e is an edge of G incident with v. Two incidences (v,e) and (w,f) of G are *adjacent* whenever (i) $v=w$, or (ii) $e=f$, or (iii) $vw=e$ or f.

An *incidence p-colouring* of G is a mapping from the set of incidences of G to the set of colours $\{1,\dots,p\}$ such that every two adjacent incidences receive distinct colours. The smallest p for which G admits an incidence p-colouring is the *incidence chromatic number* of G, denoted by $\chi_i(G)$. Incidence colourings were first introduced and studied by Brualdi and Quinn Massey [2]. Incidence colourings of various graph families have attracted much interest in recent years, see for instance [3, 4, 5, 6, 7, 8, 9].

A *strong edge p-colouring* of G is a mapping from the set of edges of G to the set of colours $\{1,\dots,p\}$ such that any two edges meeting at a common vertex, or being adjacent to a same edge of G, are assigned different colours. The smallest p for which G admits a strong edge p-colouring is the *strong chromatic index* of G, denoted by $\chi'_s(G)$.

The strong version of incidence colouring is defined in a similar way. A *strong incidence p-colouring* of a graph G is a mapping from the set of incidences of G to a finite set of colours $\{1,\dots,p\}$ such that any two incidences that are adjacent or adjacent to the same incidence receive distinct colours. The smallest p for which G admits a strong incidence p-colouring is the *strong incidence chromatic number*, denoted by $\chi_i^s(G)$.

2. PRELIMINARY RESULTS

We list in this section some basic results on the strong incidence chromatic number of various graph classes. The following observation will be useful.

Observation 1 *For every incidence (v,vu) in a graph G with maximum degree Δ, the set of incidences that are strongly adjacent to (v,vu) is*

$$\bigcup_{w\in N(v)\setminus u} A^+(w) \cup \bigcup_{w\in N(v)} A^-(w) \cup \bigcup_{w\in N(u)\setminus v} A^-(w),$$

whose cardinality is at most $3\Delta^2-2\Delta$.

Proposition 2 *For every graph G with maximum degree Δ, $\chi_i^s(G) \leq 3\Delta^2-2\Delta+1$.*

For a given graph G with maximum degree Δ, we let

$$\sigma(G) = \max_{uv\in E(G)}\{2\deg_G(v)+\deg_G(u)-1\}.$$

In the following proposition we give an upper bound on the strong incidence chromatic number of a graph G as a function of its strong chromatic index.

Proposition 3 *For every graph G, $\chi_i^s(G) \leq 2\chi_s'(G)$.*

3. SIMPLE GRAPH CLASSES

In this section, we determine the strong incidence chromatic number of stars, complete graphs, cycles, trees and wheel graphs.

We denote by $S_n,\ n\geq 1$, the star of order $n+1$, by $K_n,\ n\geq 1$, the complete graph of order n and by $K_{m,n},\ m\geq n\geq 2$, the complete bipartite graph with parts of size m and n. In [2], Brualdi and Massey showed that $\chi_i(S_n)=n+1,\ \chi_i(K_n)=n$ and $\chi_i(K_{m,n})=m+2$, for all $m\geq n\geq 2$. Since all incidences of any graph in these classes of graphs are pairwise strongly adjacent, we have the following proposition.

Proposition 4

1. *For every $n\geq 1$, $\chi_i^s(S_n)=2n$,*
2. *for every $n\geq 2$, $\chi_i^s(K_n)=2|E(G)|$,*
3. *for every $m\geq n\geq 2$, $\chi_i^s(K_{m,n})=2nm$.*

Let C_n, $n\geq 3$, denote the cycle of order n.

Theorem 5 *Let n be a positive integer such that $n\geq 4$ and $2n=5q+r$, with $q>0$ and $0\leq r\leq 4$. Then $\chi_i^s(C_n)=5+\lceil r/q\rceil$.*

We now determine the value of $\chi_i^s(W_n)$, where $W_n,\ n\geq 3$, is the wheel graph of order $n+1$, obtained from C_n by adding a universal vertex.

Theorem 6 *Let n be a positive integer such that $n\geq 3$ and $2n=5q+r$, with $q>0$ and $0\leq r\leq 4$. Then $\chi_i^s(W_n)=5+2n+\lceil r/q\rceil$.*

We finally determine the strong incidence chromatic number of trees.

Theorem 7 *If G is a tree then* $\chi_i^s(G) = \max_{uv\in E(G)} \{2\deg_G(v) + \deg_G(u) - 1\} = \sigma(G)$.

The *ladder graph*, denoted by L_h, is obtained from two paths of order h, $h \geq 1$, $P_h = v_1 \dots v_h$ and $P'_h = v'_1 \dots v'_h$ by adding the edges $v_i v'_i$, $1 \leq i \leq h$. In the following theorem, we give the value of $\chi_i^s(L_h)$.

Theorem 8 *For every integer $h \geq 3$, $\chi_i^s(L_h) = 10$.*

Recall first that a *Halin graph H* is a planar graph obtained from a tree of order at least 4 with no vertex of degree 2, by adding a cycle connecting all its leaves [**?**]. We call this cycle the *outer cycle of H*. The subgraph T obtained by deleting all the edges of the outer cycle of H is thus a tree, called the *internal tree of H*.

4. SUBCLASSES OF HALIN GRAPHS

In this section, we determine the exact value of – or upper bounds on – the strong incidence chromatic number of every Halin graph whose internal tree is either a comb or a double star.

4.1. Halin graphs whose internal tree is a comb

A tree is called a $(3,1)$-tree if the degree of each non-leaf vertex is 3. A caterpillar is a tree T such that, after deleting all its leaves, the remaining graph is a simple path called the *spine of T*. A *comb* is a caterpillar which is also a (3,1)-tree. It is easy to see that every Halin graph whose internal tree is a comb is a cubic Halin graph. In particular, if the spine has one vertex then this is the complete graph K_4.

For every integer $h \geq 1$, we construct a Halin graph H_h of order $2h+2$ whose internal tree T_h is a comb, using the construction given in [**?**]. Let $P_h = v_1 v_2 \dots v_h$ be the spine of H_h. We denote by ℓ_1 and ℓ'_1 (resp. ℓ_h and ℓ'_h) the two leaves of v_1 (resp. v_h), by ℓ_i the unique leaf of v_i, $2 \leq i \leq h-1$, and by C_h the outer cycle of H_h.

Let $\mathscr{H}_h^c$ be the set of all Halin graphs whose internal tree is a comb of order $2h+2$. A Halin graph H_h such that $C_h = \ell'_1 \ell_1 \ell_2 \dots \ell_h \ell'_h \ell'_1$ is called a *necklace*. We denote by N_h the (unique) necklace of order $2h+2$. Observe that $\mathscr{H}_h^c = \{N_h\}$ for every h, $1 \leq h \leq 3$. We prove that if H is not a necklace then this bound can be decreased to 14.

Theorem 9 *If $H \in \mathscr{H}_h^c \setminus \{N_h\}$, $h \geq 4$, then $11 \leq \chi_i^s(H) \leq 14$.*

We now determine the value of the strong incidence chromatic number of necklaces.

Theorem 10 *For every necklaces N_h, $h \geq 1$, we have*

$$\chi_i^s(N_h) = \begin{cases} 12 & \textit{if } h = 1,2,3,5, \\ 11 & \textit{otherwise.} \end{cases}$$

4.2. Halin graphs whose internal tree is a double star

The *double star*, denoted by $S_{m,n}$, $m \geq n \geq 2$, is the graph obtained from the stars S_m and S_n by adding an edge joining the central vertex v of S_m to the central vertex u of S_n. The Halin graph $HD_{m,n}$ is the Halin graph whose internal tree is the double star $S_{m,n}$ and whose outer cycle is $u_1 \dots u_n v_m \dots v_1 u_1$. We denote by P the path $v_1 \dots v_m$ and by P' the path $u_1 \dots u_n$. It is easy to see that for every graph $HD_{m,n}$, $m \geq n \geq 2$, the incidences of the set

$$A^-(v) \cup A^+(v) \cup A^-(u) \cup \{(v_1, v_1 u_1)\},$$

of cardinality $2\deg(v) + \deg(u) = \sigma(HD_{m,n}) + 1$, are pairwise strongly adjacent. Therefore, we have the following inequality.

Proposition 11 *For every two integers m and n,* $m \geq n \geq 2$, $\chi_i^s(HD_{m,n}) \geq 2m+n+3 = \sigma(HD_{m,n})+1$.

By the following theorem, we determine the value of the strong incidence chromatic number of $\chi_i^s(HD_{m,n})$.

Theorem 12 *For every two integers m and n,* $m \geq n \geq 3$,

$$\chi_i^s(HD_{m,n}) = \begin{cases} \sigma(HD_{m,n})+4 & \text{if } n=2 \text{ and } m=2, \\ \sigma(HD_{m,n})+3 & \text{if } n=2 \text{ and } m \neq 2, \\ & \text{or } (n,m) \in \{(3,\ 3),\ (3,\ 4)\}, \\ \sigma(HD_{m,n})+2 & \text{if } n=3 \text{ and } m \notin \{3,\ 4\}, \\ & \text{or } n=4 \text{ and } m \not\equiv 3 \pmod 5, \\ \sigma(HD_{m,n})+1 & \text{otherwise.} \end{cases}$$

5. CONCLUSIONS

In this paper, we have introduced and studied the strong version of incidence colouring. We have determined the exact value of – or upper bounds on – the strong incidence chromatic number of several classes of graphs, namely cycles, wheel graphs, trees, ladder graphs and some subclasses of Halin graphs. We leave as open problems the following questions.

1. What is the best possible upper bound on the strong incidence chromatic number of graphs with bounded maximum degree ? In particular, what about graphs with maximum degree 3 ?
2. What is the best possible upper bound on the strong incidence chromatic number of Halin graphs ?
3. What is the best possible upper bound on the strong incidence chromatic number of d-degenerated graphs ?

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