COMMON FIXED POINT FOR MULTIVALUED (ψ,θ,G)-CONTRACTION TYPE MAPS IN METRIC SPACES WITH A GRAPH STRUCTURE

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ABSTRACT

In this work, we introduce the notion of a common (ψ,θ,G)-contraction multi-valued mapping in order to establish some new common fixed point theorems for these classes of mappings in complete metric spaces endowed with a graph. An example of application illustrates the main existence result. Our results generalize some recent known results.

1. INTRODUCTION

Since the proof of Banach's fixed contraction principle [2], many research works have considered different kinds of generalizations. Among them, the classical multi-valued version was established by Covitz and Nadler [11] in 1969 using the Hausdorff-Pompeiu metric in a complete metric space.

In 2008, Jachymski [7] introduced a new approach in metric fixed point theory by replacing the order structure with a graph structure on a metric space. They introduced the concept of G-contraction, that is a single-valued contraction mapping in a metric space with a graph. Afterwards, many authors extended the Banach G-contraction in different ways (we refer to [1], [3], [12], [14], [15] and references therein).

Recently, Jleli and samet [9] introduced the θ-contraction and proved a fixed point result as a generalization of the Banach contraction principle. In turn, their result has also been extended by many authors (see, e.g., [5], [6], [8], [10], [16]).

Consistent with [9], we denote by Θ the set of all functions $\theta:(0,\infty)\to(1,\infty)$ satisfying the following conditions :
(Θ_1) θ is non-decreasing,
(Θ_2) for each sequence $(t_n)_n\subset(0,\infty)$, $\lim\limits_{n\to\infty}\theta(t_n)=1$ if and only if $\lim\limits_{n\to\infty}t_n=0^+$,
(Θ_3) there exist $r\in(0,1)$ and $l\in(0,\infty]$ such that $\lim\limits_{t\to0^+}\frac{\theta(t)-1}{t^r}=l$.

The aim of this paper is to prove some common fixed point results for (ψ,θ,G)-contractions multi-valued mappings in a metric space endowed with a graph G.

First of all, we collect some basic notions and primary results we need to develop our results.

Let (X,d) be a metric space. We shall denote by $CB(X)$ the family of nonempty closed bounded subsets of X and by $C(X)$ the family of nonempty closed subsets of X. For $A,B\in C(X)$, let

$$H(A,B) = \max\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\},$$

where $d(x,B) = \inf\{d(x,y) : y \in B\}$. H is called the Hausdorff-Pompeiu distance on $C(X)$. This is a metric on $CB(X)$.

A graph G is an ordered pair (V,E), where V is a set and $E \subset V \times V$ is a binary relation on V. Elements of E are called edges and are denoted by $E(G)$ while elements of V, denoted $V(G)$, are called vertices. If the direction is imposed in E, that is the edges are directed, then we get a digraph (directed graph). Hereafter, we assume that G has no parallel edges, i.e. two vertices cannot be connected by more than one edge. Thus, G can be identified with the pair $(V(G),E(G))$. If x and y are vertices of G, then a path in G from x to y of length $k \in \mathbb{N}$ is a finite sequence $(x_n)_n$, $n \in \{0,1,2,\dots k\}$ of vertices such that $x = x_0, \dots, x_k = y$ and $(x_{n-1},x_n) \in E(G)$ for $n \in \{1,2,\dots,k\}$. A graph G is connected if there is a path between any two vertices and it is weakly connected if $\widetilde{G}$ is connected, where $\widetilde{G}$ denotes the undirected graph obtained from G by ignoring the direction of edges. Let G^{-1} be the graph obtained from G by reversing the direction of edges (the conversion of the graph G). We have

$$E(G^{-1}) = \{(x,y) \in X \times X : (y,x) \in E(G)\}.$$

It is more convenient to treat $\widetilde{G}$ as a directed graph for which the set of edges is symmetric. Then

$$E(\widetilde{G}) = E(G) \cup E(G^{-1}).$$

Let G_x be the component of G consisting of all the edges and vertices which are contained in some path in G beginning at x. If G is such that $E(G)$ is symmetric, then for $x \in V(G)$, we may define the equivalence class $[x]_G$ on $V(G)$ by the relation xRy if there is a path in G from x to y. Then $V(G_x) = [x]_G$.

Throughout this paper, (X,d) denotes a metric space, $G = (V(G),E(G))$ is a directed graph without parallel edges with $V(G) = X$ and $(x,x) \notin E(G)$ (the graph does not contain loops). The following condition first appeared in [7] :

Property (A) : for any sequence $(x_n)_n$ in X, if $x_n \to x$ and $(x_n,x_{n+1}) \in E(G)$, for all $n \in \mathbb{N}$, then $(x_n,x) \in E(G)$ for $n \in \mathbb{N}$.

With this condition, Jachymski showed that in a complete metric space, a G-contraction has a fixed if and only if $X_f \neq \emptyset$, where

$$X_f = \{x \in X : (x, f(x)) \in E(G)\}. \tag{1}$$

2. MAIN RESULT

Consider the following classes of functions :

Definition 1 *We denote by Ψ the set of functions $\psi : (1,\infty) \to (1,\infty)$ satisfying the following conditions :*
(i) ψ is non-decreasing ;
(ii) for each sequence $(t_n)_n \subset (1,\infty)$, $\lim_{n\to\infty} \psi(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 1$.

Now we give the following definition

Definition 2 *Let (X,d) be a metric space endowed with a graph G. Two mappings T_1, $T_2 : X \to C(X)$ are said to be a common (ψ,θ,G)-contraction if for all $x,y \in X$ such that $(x,y) \in E(G)$ and $a \in T_i(x)$, there exists $b \in T_j(y)$ for $i,j \in \{1,2\}$ with $i \neq j$ such that $(a,b) \in E(G)$ and*

$$\psi(\theta(d^p(a,b))) \leq \psi([\theta((M_p(T_ix,T_jy))]^{k(d(x,y))}) + LN_p(T_ix,T_jy),$$

where

$$N_p(T_ix,T_jy)=\min\{d^p(x,T_i(x)),d^p(y,T_j(y)),d^p(y,T_i(x)),d^p(x,T_j(y))\},$$

$$\begin{aligned}M_p(T_ix,T_jy) &= \max\Big\{d^p(x,y),d^p(x,T_i(x)),d^p(y,T_j(y)),\\ &\quad \frac{d^p(y,T_i(x))+d^p(x,T_j(y))}{2^p},\\ &\quad \frac{d^p(x,T_i(x))d^p(y,T_j(y))}{1+d^p(x,y)},\frac{d^p(y,T_i(x))d^p(x,T_j(y))}{1+d^p(x,y)}\Big\},\end{aligned}$$

$k:(0,+\infty)\to[0,1)$ *satisfies* $\limsup_{s\to t^+} k(s)<1$, *for all* $t\in[0,+\infty)$, $L\geq 0$, $\theta\in\Theta$, $\psi\in\Psi$, $\psi\circ\theta$ *is lower semi continuity, and* $1\leq p<\frac{1}{r}$.

Our existence results for common fixed points are collected in the following :

Theorem 1 *Let (X,d) be a complete metric space endowed with a directed graph G and suppose that the triple (X,d,G) has the property (A). Let $T_1,T_2:X\to C(X)$ be a common (ψ,θ,G)-contraction. Then the following statements hold :*
(1) For every $x\in X_{T_i}$, $i=1$ or $i=2$, the mappings T_1, $T_2\mid_{[x]_{\widetilde{G}}}$ have a common fixed point, where X_{T_i} is as defined in (1).
(2) If $X_{T_i}\neq\emptyset$, $i=1$ or $i=2$, and G is weakly connected, then T_1 and T_2 have a common fixed point in X.
(3) If $X'=\cup\{[x]_{\widetilde{G}}:x\in X_{T_i}\}$, $i=1$ or $i=2$, then T_1, $T_2\mid_{X'}$ have a common fixed point.
(4) If $Graph(T_i)\subseteq E(G)$, $i=1$ or $i=2$, then T_1 and T_2 have a common fixed point.

3. REFERENCES

[1] M.R. Alfuraidan, M. Bachar, and M.A. Khamsi, *A graphical version of Reich's fixed point theorem*, J. Nonlinear Sci. Appl. 9 (2016), no. 6, 3931-3938.

[2] S. Banach, *Sur les opérations dans les ensembles abstraits et leur applications aux équations intégrales*, Fundam. Math. 3, 133-181 (1922).

[3] F. Bojor, *Fixed point theorems for Reich type contractions on metric spaces with a graph*, Nonlinear Anal. 75, 3895-3901 (2012).

[4] G. Durmaz *Some theorems for a new type of multivalued contractive maps on metric space*, Turkish J. Math. 41 (2017), no. 4, 1092–1100.

[5] H.A. Hançer, G. Minak, and I. Altun, *On a broad category of multivalued weakly Picard operators*, Fixed Point Theory. 18 (2017), 229-236.

[6] N. Hussain, V. Parvaneh, B. Samet, and C. Vetro, *Some fixed point theorems for generalized contractive mappings in complete metric spaces*, Fixed Point Theory Appl. 2015, 2015 :185, 17 pp

[7] J. Jachymski, *The contraction principle for mappings on a metric with a graph*, Proc. Am. Math. Soc. 136 (2008), no. 4, 1359-1373.

[8] M. Jleli, E. Karapinar, and B. Samet, *Further generalizations of the Banach contraction principle*, J. Inequal. Appl. 2014, 2014 :439, 9 pp.

[9] M. Jleli, B. Samet, *A new generalization of the Banach contraction principle*, J. Inequal. Appl. 2014, 2014 :38, 8 pp.

[10] G. Minak, I. Altun, *Overall approach to Mizoguchi-Takahashi type fixed point results*, Turk. J. Math., 40 (2016), 895-904.

[11] S.B. Nadler, *Multi-valued contration mappings*, Pacific J. Math., 30, 1969, no. 2, 475-488.

[12] A. Nicolae, D. O'Regan, and A. Petrusel, *Fixed point theorems for single-valued and multivalued generalized contractions in metric spaces endowed with a graph*, Georgian Math. J. 18, 307-327 (2011).

[13] B.D. Rouhani, S. Moradi, *Common fixed point of multivalued generalized φ-weak contractive mappings*, Fixed Point Theory Appl. (2010), 13 p. doi :10.1155/2010/708984.

[14] M. Samreen, T. Kamran, *Fixed point theorems for integral G-contractions*, Fixed Point Theory Appl. 2013, 149 (2013). doi :10.1186/1687-1812-2013-149.

[15] J. Tiammee, S. Suantai, *Coincidence point theorems for graph-preserving multi-valued mappings*, Fixed Point Theory Appl. 2014, 70 (2014). doi :10.1186/1687-1812-2014-70.

[16] D.W. Zheng, Z.Y. Cai, and P. Wang, *New fixed point theorems for θ-ϕ contraction in complete metric spaces*, J. Nonlinear Sci. Appl., 10 (2017), 2662–2670.