# ON THE $L^p$ BOUNDEDNESS OF A CLASS OF SEMICLASSICAL FOURIER INTEGRAL OPERATORS

*Ouissam Elong*

Laboratory of Fundamental and Applied Mathematics of Oran (LMFAO), University of Oran1, Ahmed Ben Bella, Oran & University of Tiaret, Algeria

**ABSTRACT**

We prove $L^q \to L^r$ boundedness of a class of semiclassical Fourier integral operators defined by smooth phase function and semiclassical rough symbols on the spatial variable $x$. We also consider a spacial case of $h$-pseudodifferential operators.

## 1. INTRODUCTION

In this paper, we are concerned with the $L^p$ mapping properties of *semiclassical Fourier integral operators* in the form

$$A_h(a,\varphi)f(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\varphi(x,\xi)} a(x,\xi,h)\hat{f}_h(\xi)\,\mathrm{d}\xi, \quad f \in \mathscr{S}(\mathbb{R}^n). \tag{1.1}$$

where $\varphi$ is called the phase function, $a$ is the symbol of the $h$-FIO $A_h(a,\varphi)$ and $f \in \mathscr{S}(\mathbb{R}^n)$ is a Schwartz function. If the phase $\varphi(x,\xi) = \langle x,\xi\rangle$, the $h$-FIO $A_h$ is called *$h$-pseudodifferential operator*.

These operators arise in the construction of solution of partial differential equations as wave equations, see [11]. Hence regularity properties of these parametrices is an important question to deal with. First works have been made with $L^2$ boundedness of $h$-PDO, with smooth symbols [11, 5], the authors showed that the operator norm is independent of the semiclassical parameter $h$ if the symbol $a$ is in the Schwartz space or is independent of $h$. $L^p$ regularity of this operators was evoked by Zworski in [11] with symbols depending only on the frequency variable $\xi$, known as *Fourier multipliers*.

For $L^2$ boundedness of $h$-FIO, their systematic study began with the work of Robert [8] with admissible symbols. Earlier, Aitemrar and Senoussaoui [1] proved $L^2$-boundedness and $L^2$-compactness of a class of $h$-FIO with weighted amplitudes. Recently, in [3], we studied $L^p$ global regularity of $h$-FIO defined by weighted non-smooth symbols $a(x,\xi)$ which behave in the spatial variable $x$ like $L^p$ functions and are smooth in the $\xi$ variable.

In the same framework, the purpose of this paper is to consider symbols $a(x,\xi,h)$ in the class $L^pS^m_{\rho,\delta}$, see definition 2.1 below, that is functions $a(.,\xi,h) \in L^p$ with all derivatives on $\xi$ and smooth in the frequency variable $\xi$ dependently with $h$. We will see in that case, operator norm of $h$-PDO is dependent of $h$.

This class of symbols was considered in the works of [6] and [7] to study semiclassical Schrödinger (or wave) operators of scattering problems, and in [2] and [4] to study microlocal limits on manifolds.

## 2. DEFINITIONS OF SYMBOLS AND PHASE FUNCTIONS

In this section we introduce the classes of symbols and phase functions which will be used throughout the paper.

**Definition 2.1** *Given $m \in \mathbb{R}$, $0 \le \rho, \le 1$, $0 \le \delta \le \frac{1}{2}$, $1 \le p \le \infty$ and $0 < h \le 1$. Let $a(x,\xi,h)$ be a measurable function in $x \in \mathbb{R}^n$ and $a(x,\xi,h) \in C^\infty(\mathbb{R}^n_\xi)$, a.e. $x \in \mathbb{R}^n$. We say $a(x,\xi,h)$ belongs to the symbol class $L^pS^m_{\rho,\delta}(\mathbb{R}^n)$, if for each multi-index $\alpha \in \mathbb{N}^n$ there exists a constant $C_\alpha > 0$ such that*

$$\left\|\partial_\xi^\alpha a(\cdot,\xi,h)\right\|_{L^p(\mathbb{R}^n)} \le C_\alpha h^{-\delta|\alpha|}\langle\xi\rangle^{m-\rho|\alpha|}.$$

**Example 2.2** *Functions of type $a(x,\xi,h) = \chi\left(x, \frac{\xi}{h^\delta}\right)$, with $\chi \in C_0^\infty(\mathbb{R}^{2n})$ independent of $h$, belong to $L^pS^0_{0,\delta}$, $p \ge 1$.*

We also need to describe the type of phase functions that we will deal with.

**Definition 2.3** *Let $k \in \mathbb{N}$. A real valued function $\varphi(x,\xi) \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$ is called a phase of order $k$ if it is positively homogeneous of degree 1 in $\xi$ and for each multi-indices $\alpha$ and $\beta$, $|\alpha| + |\beta| \ge k$, there exists a constant $C_{\alpha,\beta} \ge 0$ such that*

$$\sup_{(x,\xi)\in\mathbb{R}^n\times\mathbb{R}^n\setminus\{0\}} |\xi|^{-1+|\alpha|}|\partial_\xi^\alpha \partial_x^\beta \varphi(x,\xi)| \le C_{\alpha,\beta}. \tag{2.1}$$

To prove global boundedness of $h$-FIO, we should impose a further condition on the phase function.

**Definition 2.4** *A real valued phase function $\varphi \in C^2(\mathbb{R}^n \times \mathbb{R}^n \setminus \{0\})$ is called strongly non-degenerate, if there exists a constant $C > 0$ such that*

$$\left|\det \frac{\partial^2 \varphi(x,\xi)}{\partial x_j \partial \xi_k}\right| \ge C, \quad \textit{for all} \quad (x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}. \tag{2.2}$$

The condition (2.2) is called strong non-degeneracy condition (or SNDC for short).

**Lemma 2.5 (Hausdorff-Young inequality)** *Let $p,q \in \mathbb{R}$ such that $1 \le p \le 2$ and $\frac{1}{p} + \frac{1}{q} = 1$. Suppose that $f \in L^p(\mathbb{R}^n)$. Then*

$$\left\|\hat{f}_h\right\|_{L^q(\mathbb{R}^n)} \lesssim h^{\frac{n}{q}} \|f\|_{L^p(\mathbb{R}^n)},$$

*where $\hat{f}_h$ stands for the semiclassical Fourier transform of the function $f$,*

$$\hat{f}_h(\xi) = \int_{\mathbb{R}^n} e^{-\frac{i}{h}\langle y,\xi\rangle} f(y)\,\mathrm{d}y, \quad f \in \mathscr{S}(\mathbb{R}^n).$$

## 3. SEEGER SOGGE STEIN DECOMPOSITION

This decomposition was introduced in [10] and detailed in [9]. It aims to linearise the phase $\varphi$ on the frequency variable $\xi$ using Littlewood-Paley decomposition twice. First, let $\Psi \in C_0^\infty$ be a smooth function on the closed ball $B(0,2)$ centred at the origin with radius 2 and let $\Psi_0 \in C_0^\infty$ with support $\operatorname{supp}\Psi \subset \{\xi \in \mathbb{R}^n : \frac{1}{2} \le |\xi| \le 2\}$. So, the Littlewood-Paley partition of unity is

$$\Psi_0 + \sum_{j=0}^{+\infty} \Psi_j(\xi) = 1,$$

where $\Psi_j(\xi) = \Psi(2^{-j}\xi)$. Second, we decompose each crown $\{2^{j-1} \le |\xi| \le 2^{j+1}\}$ into truncated cones

$$\Gamma_j^\nu = \left\{\xi : \left|\frac{\xi}{|\xi|} - \xi_j^\nu\right| \le 2 \cdot 2^{-\frac{j}{2}}\right\}.$$

Let $\chi_j^\nu(\xi)$, $\nu = 1, \cdots, J_j$ be homogeneous functions of degree 0 supported in the cone $\Gamma_j^\nu$, such that

$$\sum_{\nu=1}^{J_j} \chi_j^\nu(\xi) = 1 \text{ for all } \xi \neq 0 \text{ and all } j.$$

Thus,

$$|\partial_\xi^\alpha \chi_j^\nu(\xi)| \leq C_\alpha 2^{\frac{|\alpha| j}{2}} |\xi|^{-|\alpha|}. \tag{3.1}$$

To finish the Seeger Sogge Stein decomposition, we write $\mathbb{R}^n = \mathbb{R}\xi_j^\nu \oplus (\xi_j^\nu)^\perp$, that is,

$$\xi = \xi_1 \xi_j^\nu + \xi', \ \xi' = (\xi_2, \cdots, \xi_n) \text{ is orthogonal to } \xi_j^\nu.$$

Hence the derivative in the $\xi_1$ direction can be improved such as

$$|\partial_{\xi_1}^N \chi_j^\nu(\xi)| \leq C_N |\xi|^{-N}, \quad N \geq 1. \tag{3.2}$$

Constructions $\Psi_j$'s and $\chi_j^\nu$'s allow us to define the second Littlewood-Paley partition of unity

$$\Psi_0(\xi) + \sum_{j=1}^{\infty} \sum_{\nu=1}^{J_j} \chi_j^\nu(\xi) \Psi_j(\xi) = 1.$$

Hence, now, we decompose the $h$-FIO $T_h(a, \varphi)$, defined in (1.1) into a low-frequency part $A_h^0$, and high frequency parts $A_h^{j,\nu}$, $j \geq 1, \nu \in \{1, \cdots, J_j\}$, in the following way :

$$\begin{aligned}
A_h(a,\varphi) f(x) &= A_h^0 f(x) + \sum_{j=1}^{\infty} A_h^j f(x) = A_h^0 f(x) + \sum_{j=1}^{\infty} \sum_{\nu=1}^{J_j} A_h^{j,\nu} f(x) \\
&= (2\pi h)^{-n} \iint_{\mathbb{R}^{2n}} e^{\frac{i}{h}\varphi(x,\xi) - \langle y,\xi \rangle} a(x,\xi,h) \Psi_0(\xi) f(y) \, \mathrm{d}y \, \mathrm{d}\xi \\
&\quad + (2\pi h)^{-n} \sum_{j=1}^{\infty} \sum_{\nu=1}^{J_j} \iint_{\mathbb{R}^{2n}} e^{\frac{i}{h}\varphi(x,\xi) - \langle y,\xi \rangle} a(x,\xi,h) \Psi_j(\xi) \chi_j^\nu(\xi) f(y) \, \mathrm{d}y \, \mathrm{d}\xi.
\end{aligned}$$

Otherwise, set $\Phi(x,\xi) = \varphi(x,\xi) - \langle \nabla_\xi \varphi(x, \xi_j^\nu), \xi \rangle$. Then

$$\Phi(x,\xi) = \langle \nabla_\xi \varphi(x,\xi) - \nabla_\xi \varphi(x, \xi_j^\nu), \xi \rangle,$$

in view of Euler's homogeneity formula. Furthermore, for $N \geq 1$,

$$|\partial_{\xi_1}^N \Phi(x,\xi)| \leq C_N \, 2^{-Nj}, \tag{3.3}$$

and

$$|(\nabla_{\xi'})^N \Phi(x,\xi)| \leq C_N \, 2^{-\frac{Nj}{2}}. \tag{3.4}$$

In another hand, put

$$a_j^\nu(x,\xi,h) = e^{\frac{i}{h}\Phi(x,\xi)} a(x,\xi,h) \chi_j^\nu(\xi) \Psi_j(\xi). \tag{3.5}$$

Using these last notations, we can write $A_h^{j,\nu}$ as an $h$-FIO with a linear phase function in $\xi$ as

$$A_h^{j,\nu} f(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle (\nabla_\xi \varphi)(x,\xi_j^\nu), \xi \rangle} a_j^\nu(x,\xi,h) \hat{f}_h(\xi) \, \mathrm{d}\xi, \quad f \in \mathscr{S}(\mathbb{R}^n). \tag{3.6}$$

The following Lemma is a straightforward application of Leibniz rule.

**Lemma 3.1** *Let $a_j^\nu$ be defined as in* (3.5) *where $a \in L^p S^m_{\rho,\delta}(\mathbb{R}^n)$ and $\varphi$ is a phase function of order* 2. *Then for every multi-index $\alpha = (\alpha_1, \alpha') \in \mathbb{N} \times \mathbb{N}^{n-1}$,*

$$\left\| \partial_\xi^\alpha a_j^\nu(\cdot,\xi,h) \right\|_{L^p(\mathbb{R}^n)} \leq C_\alpha \, h^{-|\alpha|(1+\delta)} \, 2^{(m-|\alpha|\rho+|\alpha'|/2)j}, \ \xi \in \mathbb{R}^n, \ 0 < h \leq 1.$$

## 4. MAIN RESULT

In this section we show that any $h$-FIO of type (1.1) with symbol in the class $L^pS^m_{\rho,\delta}$ and phase function of order 2 is $L^q \to L^r$-bounded under some conditions on $m, p$ and $q$ and. Using decomposition (3.3), we shall prove first the $L^q \to L^r$-boundedness of $h$-FIO $A_h^0$ with compact support symbol in the frequency variable $\xi$ and thenceforth we move to prove this $L^q \to L^r$-boundedness of the high frequency term $A_h^j$.

**Theorem 4.1** *Let $0 \le \rho, \le 1$, $0 \le \delta \le \frac{1}{2}$, $0 < h \le 1$, $m \in \mathbb{R}$, $0 < r \le \infty$ and $1 \le p, q \le \infty$ such that $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Assume that $a_0 \in L^pS^m_{\rho,\delta}(\mathbb{R}^n)$ and $\operatorname{supp}_\xi a_0(x,\xi,h)$ is compact and $\varphi$ is a phase function of order 2 satisfying the SNDC (2.2). Then*

$$\|A_h(a_0,\varphi)\|_{L^q\to L^r} \lesssim h^{-N\delta}, \quad N = \max\left\{n+m, \frac{n}{r}\right\} + 1.$$

We prove now the boundedness of the high frequency part.

**Theorem 4.2** *Let $0 \le \rho, \le 1$, $0 \le \delta \le \frac{1}{2}$, $0 < h \le 1$, $0 < r \le \infty$, $1 \le p, q \le \infty$ verify $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Suppose that $\varphi$ is a phase function of order 2 satisfying the SNDC (2.2) and assume $a \in L^pS^m_{\rho,\delta}(\mathbb{R}^n)$ such that, for some $\varepsilon > 0$,*

$$m < \frac{\rho n}{s} - 2M - \frac{n-1}{2}\left(\frac{1}{s} + \frac{1}{\min(p,s')}\right), \quad C_0 > 0, \quad \xi \in \mathbb{R}^n, \tag{4.1}$$

*with $s = \min(2,p,q)$, $\frac{1}{s} + \frac{1}{s'} = 1$, and some $M > \frac{n}{2s}$. Then the h-FIO $A_h(a,\varphi)$ defined in (1.1) is bounded from $L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ and*

$$\|A_h(a,\varphi)\|_{L^q\to L^r} \le C h^{-2M(1+\delta)}, \quad C > 0.$$

Note that if we study pseudodifferential operators (PDOs)

$$a(x,hD)f(x) = (2\pi h)^{-n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x,\xi\rangle} a(x,\xi,h) \hat{f}_h(\xi)\,d\xi,\ f \in \mathscr{S}(\mathbb{R}^n), \tag{4.2}$$

the phase $\langle x,\xi\rangle$ is linear in $\xi$. In this case the Seeger-Sogge-Stein decomposition is not necessary to prove the global $L^q \to L^r$ boundedness and it suffices to use just the Littlewood-Paley decomposition. Thus we have the following result.

**Theorem 4.3** *Let $0 \le \rho, \le 1$, $0 \le \delta \le \frac{1}{2}$ $0 < h \le 1$, $0 < r \le \infty$ and $1 \le p, q \le \infty$ verify $\frac{1}{r} = \frac{1}{p} + \frac{1}{q}$. Assume that $a \in L^pS^m_{\rho,\delta}(\mathbb{R}^n)$ with*

$$m < \frac{n(\rho-1)}{\min(2,p,q)}. \tag{4.3}$$

*Then the h-PDO $a(x,hD)$ is bounded from $L^q(\mathbb{R}^n)$ to $L^r(\mathbb{R}^n)$ and*

$$\|a(x,hD)\|_{L^q\to L^r} \le h^{-2\delta M},$$

*with $M > \frac{n}{2\min(2,p,q)}$.*

## 5. REFERENCES

[1] C. A. Aitemrar and A. Senoussaoui. *h-admissible Fourier integral operators*. *Turkish J. Math.*, 40(3) :553–568, 2016.

[2] S. Dyatlov and C. Guillarmou. Microlocal limits of plane waves and Eisenstein functions. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(2) :371–448, 2014.

[3] O. Elong and A. Senoussaoui. *On the $L^p$-boundedness of a class of semiclassical Fourier integral operators. Matematicki Vesnik Journal*, 70(3) :189–203, 2018.

[4] S. Eswarathasan and S. Nonnenmacher. Strong scarring of logarithmic quasimodes. *Ann. Inst. Fourier (Grenoble)*, 67(6) :2307–2347, 2017.

[5] A. Martinez. *An introduction to semiclassical and microlocal analysis.* New York, NY : Springer, 2002.

[6] S. Nonnenmacher. Spectral problems in open quantum chaos. *Nonlinearity*, 24, 2011.

[7] S. Nonnenmacher, J. Sjöstrand, and M. Zworski. Fractal Weyl law for open quantum chaotic maps. *Ann. of Math. (2)*, 179(1) :179–251, 2014.

[8] D. Robert. *Autour de l'approximation semi-classique. (Around semiclassical approximation).* Birkhäuser, Springer, Basel, 1987.

[9] S. Rodríguez-López and W. Staubach. *Estimates for rough Fourier integral and pseudodifferential operators and applications to the boundedness of multilinear operators. J. Funct. Anal.*, 264(10) :2356–2385, 2013.

[10] A. Seeger, C. D. Sogge, and E. M. Stein. *Regularity properties of Fourier integral operators. Ann. Math. (2)*, 134(2) :231–251, 1991.

[11] M. Zworski. *Semiclassical analysis.* Providence, RI : American Mathematical Society (AMS), 2012.