SELF-SIMILAR SOLUTIONS FOR FREE-BOUNDARY PROBLEM FROM CONTOUR ENHANCEMENT IN IMAGE PROCESSING

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ABSTRACT

Many models which use non-linear PDEs have been extensively used for different tasks of edge enhancement in image processing, based on a new evolution model consider as a generalization of mean curvature motion. A free boundary problem is formulated describing the image intensity evolution in the boundary layers around the edges of image. An asymptotic self-similar solutions to this nonlinear diffusion equation are obtained in explicit forms.

1. INTRODUCTION

For the past decades, methods based on partial differential equations (PDEs) have been extensively used to model processes in numerous real-world applications, ranging from physics over life sciences to economy. Thus, it is not surprising that many partial differential equations (PDEs) and variational approaches have been contributed substantially to the mathematical foundations of signal and image analysis. For instance, they appear as Euler-Lagrange equations when solving continuous optimization problems that result from variation models [2, 6] or regularizations of ill-posed problems [5]. It has also been shown that they are the natural setting for scale-spaces [1], they are successfully used for image enhancement [11]. Usually the two prominent directions for the local geometry in the image are the direction of the level or isophote (along an edge) and its orthogonal direction, the flow-line (across an edge). As a result of a certain degeneracy of the asymptotic forms of equations mentioned in [9, 10], Barenblatt [3] noted the possibility to construct a more general class of equations which generalized mean curvature motion and Beltrami flow. An asymptotic treatment in one dimensional case of this class of equations has been investigated theoretically in [3, 4] and [7, 8]. Our goal in this study is to present a new model which generalize the mean curvature motion, in order to focus on the phenomenon of contour enhancement. An asymptotic treatment in one dimensional case will be discussed by using different types of self-similar solutions.

2. GENERALIZED MEAN CURVATURE MOTION

A common denoising technique is to minimize a functional of gradient given as :

$$\min_u \mathscr{F}_\alpha (u), \ \mathscr{F}_\alpha (u) = \int_\Omega |\nabla u|^\alpha \, dx, \tag{1}$$

where $\mathscr{F}_1 (u)$ is called the total variation flow (TV flow).
In solving (1) , it is often convenient to transform the minimization problem into a differential equation, called the Euler-Lagrange equation :

$$\frac{\partial u}{\partial t} = div \left(\frac{\nabla u}{|\nabla u|^{2-\alpha}} \right), \tag{2}$$

Mean curvature motion can be tackled by the use of the level set method [**?**, **?**], which consists in viewing the curve as the level set $u = 0$ of a function u constrained to solve the degenerate diffusion equation :

$$u_t = |\nabla u|\, div\left(\frac{\nabla u}{|\nabla u|}\right). \tag{3}$$

See [1] for the MMC and related work. Combining TV flow and MMC, we consider a generalization :

$$\frac{\partial u}{\partial t} = |\nabla u|^q\, div\left(\frac{\nabla u}{|\nabla u|^p}\right). \tag{4}$$

Which we call the generalized mean curvature motion in this article.
The constants p and q plays the role of enhancement parameters, because they weights the rate at which diffusion vanishes as $|\nabla u|$ grows up. Equation (4) is regard as a generalisation of mean curvature motion and the TV flow. In more literal formulation, equation(4) can also be written :

$$\frac{\partial u}{\partial t} = \frac{u_{xx}\left[u_y^2 + (1-p)\,u_x^2\right] + u_{yy}\left[u_x^2 + (1-p)\,u_y^2\right] - 2pu_xu_yu_{xy}}{\left(u_x^2 + u_y^2\right)^{\frac{p}{2}-\frac{q}{2}+1}}. \tag{5}$$

In order to focus of the contour enhancement it is appropriate to flow the asymptotic analysis performed by Barenblatt [3]. In this way, equation (5) in the boundary layer is reduced to the one-dimensional form :

$$u_t\,(x,t) = (1-p)\,u_x^{q-p}u_{xx}. \tag{6}$$

Equation (6) governing the evolution of the image intensity in the boundary layer.

3. FREE BOUNDARY PROBLEM TO CONTOUR ENHANCEMENT

We introduce the following free boundary problem for determining the image intensity evolution in the boundary layer : Given an increasing function $u_0\,(x)$ defined in an interval $I = (a_1, a_2)$ with end values $u_0\,(a_1) = 0$, $u_0((a_2) = 1$, to find a continuous function $u\,(x,t)$ and continuous curves $x = l\,(t)$ and $x = r\,(t)$ such that :

1. $l\,(0) = a_1$, $r\,(0) = a_2$, and $l\,(t) < r\,(t)$ for some time interval $t \in (0,T)$.
2. u solves the following problem in $\Omega = \{(x,t) : 0 < t < T,\ l\,(t) < x < r\,(t)\}$:

$$\begin{cases} \frac{\partial u}{\partial t} = (1-p)\,u_x^{q-p}u_{xx}, & \text{in } \Omega \\ u((x,0) = u_0\,(x), & \text{for } a_1 \le x \le a_2 \\ u(l\,(t)\,,t) = 0, u_x(l\,(t)\,,t) = +\infty & \text{for } 0 < t < T \\ u(r\,(t)\,,t) = 1, u_x(r\,(t)\,,t) = +\infty & \text{for } 0 < t < T \end{cases}, \tag{7}$$

for $q < p < 1$ and $q > p > 1$. T can be finite or infinite time.
The mathematical theory for the free-boundary problem (7) was treated in [4]. The solution $u\,(x,t)$ sharpens as t grows ; actually the two free boundaries $x = l\,(t)$ and $x = r\,(t)$ shrink. If the two moving boundaries meet, a vertical front is formed, representing completed enhancement see(Figure 1).

4. SELF SIMILAR SOLUTIONS

We try now to solve problem (7) by introducing the the self-similar solution written under the form :

$$u\,(x,t) = \phi\,(\xi) \quad, \xi = \frac{x - x_0}{a\,(t)}. \tag{8}$$

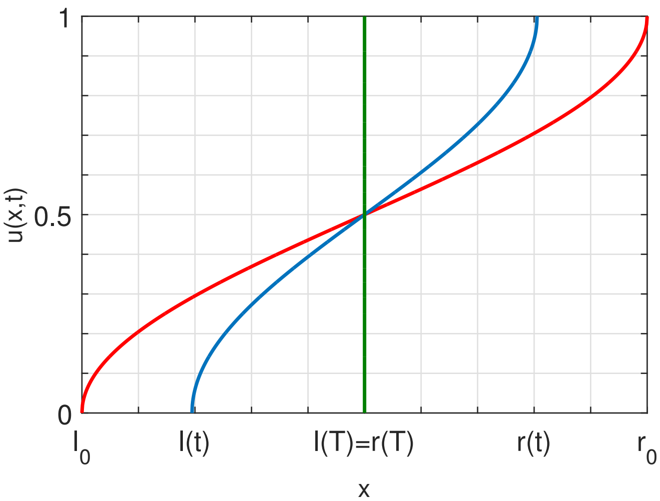

FIGURE 1 – At the left side, the front of the solution to the free boundary problem, respectively at time 0, after some time t, and when complete enhancement occurs.

Taking account this form, Substituting (8) into the equation (6), we obtain :

$$\frac{d\phi}{d\xi} = \left[\frac{-\alpha\,(q-p)}{2\,(1-p)}\right]^{\frac{1}{q-p}} C^{\frac{2}{q-p}} \left[1-\left(\frac{\xi}{C}\right)^2\right]^{\frac{1}{q-p}}, \tag{9}$$

The problem (7) suggests a free-boundary problem for determination of the image intensity evolution in the boundary layer $l\,(t)$ and $r\,(t)$ such that $l\,(t) \leq x \leq r\,(t)$, with $l\,(t) = x_0 - Ca\,(t)$ and $r\,(t) = x_0 + Ca\,(t)$.

Further integration of (9) and using the boundary conditions, implies $\phi\,(-C) = 0$, $\phi\,(C) = 1$, then we obtain :

$$\phi\,(\xi) = \left[\frac{-\alpha\,(q-p)}{2\,(1-p)}\right]^{\frac{1}{q-p}} C^{\frac{q-p+2}{q-p}} \int_{-1}^{\frac{\xi}{C}} \left[1-\eta^2\right]^{\frac{1}{q-p}} d\eta,$$

where

$$C = \left[\frac{-\alpha\,(q-p)}{2\,(1-p)}\right]^{-\frac{1}{q-p+2}} \left[2\int_0^1 \left[1-\eta^2\right]^{\frac{1}{q-p}} d\eta\right]^{-\frac{q-p}{q-p+2}},$$

for $q \neq p-2$.

As mentioned above, different values of p and q lead to different behaviours of self-similar solutions : the large time behaviour depends on the parameters p and q.

We studying the case $q < p-2$ with $p < 1$. Note that, in this case, solutions are defined globally in time. The self-similar solution assumes the form

$$u\,(x,t) = \left[\frac{-\alpha\,(q-p)}{2\,(1-p)}\right]^{\frac{1}{q-p}} C^{\frac{q-p+2}{q-p}} \int_{-1}^{\frac{x-x_0}{Ca(t)}} \left[1-\eta^2\right]^{\frac{1}{q-p}} d\eta, \tag{10}$$

and its gradient is given by

$$u_x\,(x,t) = \frac{1}{a(t)} \left[\frac{-\alpha\,(q-p)}{2\,(1-p)}\right]^{\frac{1}{q-p}} C^{\frac{2}{q-p}} \left[1-\left(\frac{x-x_0}{Ca(t)}\right)^2\right]^{\frac{1}{q-p}}.$$

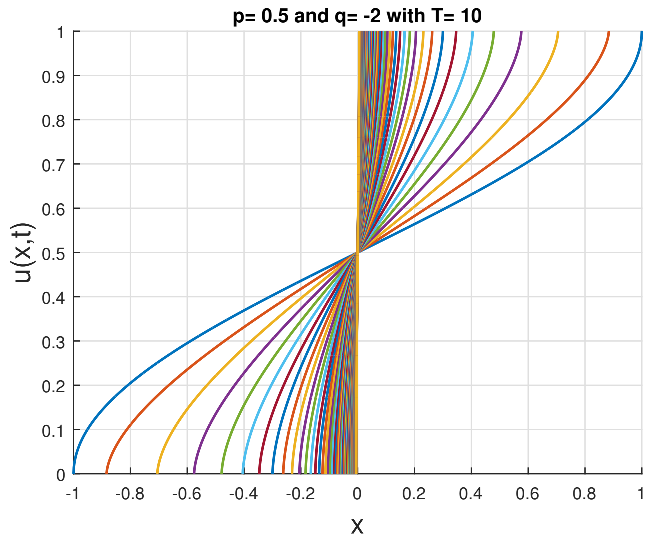

FIGURE 2 – Self-similar solutions of type I for different values of the parameters $p < 1$ and $q < p - 2$.

It is seen that, solution (10) is a local solution. At free boundaries $x = l(t)$ and $x = r(t)$, the image intensity is continuous but the derivative u_x suffers an infinite jump.

5. NUMERICAL EXPERIMENTS

We have computed the self-similar solutions of the free boundary problem (7) for different values of parameters p and q. The computations have been done with symmetrical smoothed initial condition u_0.
In Figure 2, the evolution of the image intensity distribution in time is presented for a monotonic initial condition for different values of p and q, in the case $q < p - 2$ with $p < 1$.
We see that across the two delimiting curves the intensity function u is continuous, but its derivative u_x suffers an infinite jump. Moreover, for $q \longrightarrow p - 2$ the evolution arrives at a vertical front rapidly, so the enhancement happens after a few steps, while for values of q much smaller than $p - 2$, the enhancement needs a larger time.

6. CONCLUSIONS

We have established the asymptotic analysis of a free boundary problem that represents a one-dimensional version of a new model consider as a generalization of mean curvature motion. Analysis of self-similar solutions for the image evolution in the boundary layer demonstrated that the edge enhancement and its rates depends on the parameters p and q.
In this model, the equation is forward diffusion for p < 1, In this case, the model has been verified to be effective for the edge enhancement only when q < p, the results demonstrated that the edge enhancement takes place.

7. REFERENCES

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