OPTIMAL DECAY FOR ABSTRACT SECOND-ORDER EVOLUTION EQUATION WITH INFINITE MEMORY AND TIME-VARYING DELAY IN HILBERT SPACES.

Houria Chellaoua,Yamna Boukhatem

Laboratory of Pure and Applied Mathematics,
University of Laghouat, Algeria,
E-mail: houriachellaoua@gmail.com.
Laboratory of Pure and Applied Mathematics,
University of Laghouat, Algeria,
E-mail: y.boukhatem@lagh-univ.dz.

ABSTRACT

In this paper, we study a second-order abstract evolution equation in Hilbert spaces with infinite memory, time-varying delay and a much larger class of kernel functions. Firstly, we prove well-posedness of solution by using the semigroup arguments and variable norm technique of Kato. Then, we establish an explicit and general decay results of the energy solution by introducing a suitable Lyaponov functional and some properties of the convex functions. This work improves the previous results with finite memory to infinite memory and without time delay/constant delay term to those with time-varying delay.

1. INTRODUCTION

Let H be a real Hilbert space with inner product and related norm denoted by $\langle .,. \rangle$ and $\| . \|$, respectively. Let $A : D(A) \longrightarrow H$ and $B : D(B) \longrightarrow H$ be self-adjoint linear positive operators with domains $D(A) \subset D(B) \subset H$ such that the embeddings are dense and compact. $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is the kernel of the memory term and $\tau(t) > 0$ represents the time-varying delay.

In this work, we consider the following second-order abstract evolution equation

$$\begin{cases} u_{tt}(t) + Au(t) - \int_0^{+\infty} h(s)Bu(t-s)ds + \mu_1 u_t(t) + \mu_2 u_t(t-\tau(t)) = 0, & t \in (0,+\infty), \\ u_t(t-\tau(0)) = f_0(t-\tau(0)) & t \in (0,\tau(0)), \\ u(-t) = u_0(t), \quad u_t(0) = u_1, & t \geq 0, \end{cases} \tag{1}$$

where the initial datum (u_0, u_1, f_0) belongs to a suitable spaces, μ_1 is a positive constant and μ_2 is a real number.

Time delays arises in many applications and practical problems and in many cases, even small delay may destabilize a system which is asymptotically stable in the absence of delay, in this sense, see [16, 5, 34]. A large part in the literature is available addressing the stability, instability and the connection between the memory term, the frictional damping and the delay terms. In particular, for wave equation with constant or variable delay, we refer to read [3, 34, 35]. They showed that the frictional damping term is strong enough to stabilize the system when the weight of the delay be sufficiently small. Nicaise and al in [36] studied a wave equation in one-space dimension and proved an exponential stability result under the condition $0 < |\mu_2| \leq \sqrt{1-d}\mu_1$, where μ_2 is a real number and the constant d satisfies $\tau'(t) \leq d < 1$, for all $t > 0$. In the absence of delay ($\mu_2 = 0$), Dafermos in [15] studied the system (1) where $\mu_1 = 0$. He showed that the energy tends asymptotically to zero, but there no decay rate is given, see [29, 30, 17, 37] and references therein. Messaoudi in [27] given a general stability where the exponential and the polynomial decay rates are special cases. Precisely, he considered kernel functions satisfy

$$h'(t) \leq -\zeta(t)h(t), \quad \forall t \in \mathbb{R}_+, \tag{2}$$

where ζ is a nonincreasing positive differentiable function. After that, Alabau Boussouira et al. [1] introduced the following condition $h'(t) \leq -G(h(t))$ where G is a convex function which appeared in many papers, see also [10, 26, 39, 31]. In [33], Mustafa established an optimal explicit and general decay results when the kernel function h satisfy

$$h'(t) \leq -\zeta(t)G(h(t)), \quad \forall t \in \mathbb{R}_+, \tag{3}$$

where G is an increasing and convex function. For some works used (3), we refer to read [28, 6, 22]. In the case of finite memory with delay, Kirane and Said-Houari in [25], considered the following wave equation

$$u_{tt}(t) - \Delta u(t) + \int_0^t h(t-s)\Delta u(s)ds + \mu_1 u_t(t) + \mu_2 u_t(t-\tau) = 0,$$

where μ_1 and μ_2 are positive constants. They established the energy decay under the condition $\mu_2 \leq \mu_1$ in the case of (2), see also [7, 8, 18, 12]. In [14], Chellaoua, Boukhatem and Feng considered a second-order abstract viscoelastic equation in Hilbert spaces with delay term in the nonlinear internal damping and a nonlinear source term. Under some suitable assumptions on the weight of the delayed feedback and the weight of the non-delayed feedback, the authors established two explicit and general decay rate results of the energy with kernel functions satisfy (3).

In the case of infinite memory with delay, Guesmia in [19] established the exponential stability and proves that the unique dissipation given by the memory term is strong enough to stabilize exponentially the following system

$$u_{tt}(t) + Au(t) - \int_0^{+\infty} h(s)Au(t-s)ds + \mu u_t(t-\tau(t)) = 0, \quad t \geq 0.$$

In [20], the same author and Tatar considered an abstract hyperbolic equations with infinite memory and distributed time delay. They given the decay rate of solutions explicitly in terms of the growth at infinity of the infinite memory and the distributed time delay, see also [11]. Recently, there is a different results according to the general decay for a several problems with internal or boundary feedback and for constant or variable delay. For instance, Mustafa in [32] considered a plate equation with infinite memory in the presence of nonlinear feedbacks with and without delay. Under the condition $h'(t) \leq -G(h(t))$. He established an explicit and general decay rate result without imposing restrictive assumptions on the behavior of the relaxation function at infinity. In [9], the same result extended to a variable coefficient viscoelastic equation with a time-varying delay in the boundary feedback and acoustic boundary conditions and nonlinear source term by Boukhatem and Benabderrahmane. They established the decay results where the kernel memory satisfies the equation (2) and

$$\int_0^{+\infty} \frac{h(s)}{G^{-1}(-h'(s))}ds + \sup_{s\in\mathbb{R}_+} \frac{h(s)}{G^{-1}(-h'(s))} < +\infty,$$

where G is an increasing positive strictly convex function satisfying some additional properties. For the general decay results where h satisfy (3), Guesmia in [21] considered the following two problems

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^{+\infty} h(s)\Delta u(t-s)ds = 0, \quad \text{in } \Omega \times \mathbb{R}_+^* \tag{4}$$

and

$$u_{tt}(x,t) - \Delta u(x,t) - \int_0^{+\infty} h(s)u(t-s)ds = 0, \quad \text{in } \Omega \times \mathbb{R}_+^*, \tag{5}$$

where Ω is a bounded domain of $\mathbb{R}^N$, $N \in \mathbb{N}^*$. The author provided a relation between the decay rate of the solutions and the growth of h at infinity under the very general assumption on h. Moreover, the boundedness assumptions on the history data have been dropped in this work. See also [2]. Chellaoua and Boukhatem in [13], established the same result of stability for the abstract problem (1) where the delay is constant $\tau(t) = \tau$.

In this work, we are interested in giving an optimal explicit and a general decay rates of the solution of the problem (1) under some assumptions on the function h, μ_1 and the weight of delay μ_2. More precisely, we are intending to extend some general decay results to the case of abstract evolution equation with infinite memory and time-varying delay in Hilbert spaces; the system (1) where the relaxation function satisfies a wider class of relaxation functions. Moreover, the boundedness assumptions have been dropped on the history data considered in many earlier results in the literature. To the best of our knowledge, there is no decay result for problems with time-varying delay and infinite memory where the kernel functions satisfy (3). Moreover, our problem generalizes the earlier problems without time delay/constant delay term to those with time-varying delay and those with finite memory to those with infinite memory.

The paper is organized as follows. In Sect. 2, we prove the well-posedness by using the semigroup arguments under suitable hypothesis. In Sect. 3, we present some technical lemmas needed for our work. Then, we establish the decay results of the energy by using the energy method to produce a suitable Lyapunov functional.

2. WELL-POSEDNESS

In this section, we state and prove the well-posedness result of problem (1). For this purpose, we define the variables z and η^t by, respectively

$$z(\rho,t) = z(x,\rho;t) = u_t(t-\rho\tau(t)), \quad \rho \in (0,1),\ t > 0.$$

$$\eta^t(s) = \eta^t(x,s) = u(x,t) - u(x,t-s), \quad t,s > 0.$$

Therefore, problem (1) takes the form

$$\begin{cases} u_{tt}(t) + Au(t) - h_0 Bu(t) + \int_0^{+\infty} h(s)B\eta^t(s)ds + \mu_1 u_t(t) & \\ \qquad +\mu_2 z(1,t) = 0, & t > 0, \\ \tau(t)z_t(\rho,t) + (1-\rho\tau'(t))z_\rho(\rho,t) = 0, & \rho \in (0,1),\ t > 0, \\ \eta_t^t(s) = u_t(x,t) - \eta_s^t(x,s), & ;t,s > 0, \\ \eta_t^t(s) = u_t(t) - \eta_s^t(s), & t,s > 0, \\ z(\rho,0) = f_0(-\rho\tau(0)), & \rho \in (0,1), \\ z(0,t) = u_t(t), & t \geq 0, \\ u(-t) = u_0(t), \quad u_t(0) = u_1, & t \geq 0, \\ \eta^0(s) = u_0(0) - u_0(s), & s > 0. \end{cases} \tag{6}$$

Let us consider the following assumptions :
(A1) There exist positive constants a and b satisfying

$$b\|u\|^2 \leq \left\|B^{\frac{1}{2}}u\right\|^2 \leq a\left\|A^{\frac{1}{2}}u\right\|^2, \quad \forall u \in D(A^{\frac{1}{2}}). \tag{7}$$

(A2) The function $\tau \in W^{2,\infty}([0,T])$, for all $T > 0$ such that

$$0 < \tau_0 \leq \tau(t) \leq \tau_1, \quad \forall t > 0, \tag{8}$$

$$\tau'(t) \leq d < 1, \quad \forall t > 0, \tag{9}$$

where τ_0 and τ_1 are two positive constants.

(A3) The coefficients of delay and dissipation satisfy

$$|\mu_2| \leq \frac{2(1-d)}{2-d}\mu_1. \tag{10}$$

(A4) The kernel function $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is of class C^1 nonincreasing function satisfying

$$h(0) > 0, \quad h_0 = \int_0^{+\infty} h(s)ds < \frac{1}{a}. \tag{11}$$

Let us denote $U = (u, u_t, \eta^t, z)^T$, the problem (6) can be rewritten :

$$\begin{cases} U_t(t) = \mathscr{A}(t)U(t), \quad \forall t > 0, \\ U(0) = U_0 = \left(u_0, u_1, \eta^0, f_0(-\tau(0).)\right)^T, \end{cases} \tag{12}$$

where the operator $\mathscr{A}(t)$ and the domain $D(\mathscr{A}(t))$ are respectively given by

$$\mathscr{A}(t)\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = \begin{pmatrix} \phi_2 \\ -(A-h_0B)\phi_1 - \int_0^{+\infty} h(s)B\phi_3(s)ds - \mu_1\phi_2 - \mu_2\phi_4(1) \\ \phi_2 - \dfrac{\partial \phi_3}{\partial s} \\ \dfrac{\tau'(t)\rho - 1}{\tau(t)}\dfrac{\partial \phi_4}{\partial \rho} \end{pmatrix},$$

with domain

$$\mathscr{D}(\mathscr{A}(t)) = \left\{ \begin{array}{c} (\phi_1, \phi_2, \phi_3, \phi_4)^T \in \mathscr{H},\ (A-h_0B)\phi_1 + \int_0^{+\infty} h(s)B\phi_3(s)ds \in H, \\ \phi_2 \in D(A^{\frac{1}{2}}),\ \dfrac{\partial \phi_3}{\partial s} \in L_h^2(\mathbb{R}_+, V), \\ \dfrac{\partial \phi_4}{\partial \rho} \in L^2(0,1;H),\ \phi_3(0) = 0,\ \phi_4(0) = \phi_2, \end{array} \right\}$$

where

$$\mathscr{H} = D(A^{\frac{1}{2}}) \times H \times L_h^2(\mathbb{R}_+, D(B^{\frac{1}{2}})) \times L^2(0,1;H).$$

The sets $L_h^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))$ and $L^2(0,1;H)$ are respectively defined by

$$L_h^2(\mathbb{R}_+, D(B^{\frac{1}{2}})) = \left\{ \phi : \mathbb{R}_+ \to D(B^{\frac{1}{2}}), \quad \int_0^{+\infty} h(s) \left\| B^{\frac{1}{2}}\phi(s) \right\|^2 ds < +\infty \right\},$$

equipped with the inner product

$$\langle \phi_1, \phi_2 \rangle_{L_h^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))} = \int_0^{+\infty} h(s) \left\langle B^{\frac{1}{2}}\phi_1(s), B^{\frac{1}{2}}\phi_2(s) \right\rangle ds.$$

And

$$L^2(0,1;H) = \left\{ \phi : (0,1) \to H, \quad \int_0^1 \|\phi(\rho)\|^2 d\rho < +\infty \right\},$$

equipped with the inner product

$$\langle \phi_1, \phi_2 \rangle_{L^2(0,1;H)} = \int_0^1 \langle \phi_1(\rho), \phi_2(\rho) \rangle d\rho.$$

The Hilbert space $\mathscr{H}$ equipped with the following usual inner product.
For all $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$ and $W = (w_1, w_2, w_3, w_4)^T$ in $\mathscr{H}$, we have

$$\begin{aligned} \langle \Phi, W \rangle_{\mathscr{H}} \quad = \quad & \langle \phi_1, w_1 \rangle_{D(A^{\frac{1}{2}})} - h_0 \langle \phi_1, w_1 \rangle_{D(B^{\frac{1}{2}})} + \langle \phi_2, w_2 \rangle + \langle \phi_3, w_3 \rangle_{L_h^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))} \\ & + \langle \phi_4, w_4 \rangle_{L^2(0,1;H)}, \end{aligned}$$

Remark 1 *Observe that the domain of $\mathscr{A}(t)$ is independent of the time t, so*

$$\mathscr{D}(\mathscr{A}(t)) = \mathscr{D}(\mathscr{A}(0)), \qquad \forall t > 0. \tag{13}$$

A general theory for equations of type (12) has been developed by using semigroup theory [23, 24, 38]. The simplest way to get the existence and uniqueness results of solution of (12) is to prove that the triplet $\{\mathscr{A}, \mathscr{H}, Y\}$, with $\mathscr{A} = \{\mathscr{A}(t) : t \in [0,T]\}$ for some fixed $T > 0$ and $Y = \mathscr{D}(\mathscr{A}(0))$, forms a constant domain system ; see [23, 24].

Theorem 1 *Assume the following :*

- **(i)** *For all $t \in [0,T]$, $\mathscr{A}(t)$ generates a strongly continuous semigroup on $\mathscr{H}$, and the family $\mathscr{A} = \{\mathscr{A}(t) : t \in [0,T]\}$ is stable with stability constants independent of t.*
- **(ii)** *Equation (13) holds.*
- **(iii)** *$\partial_t \mathscr{A}$ belongs to $L^\infty_*([0,T], B(Y, \mathscr{H}))$ the space of equivalent classes of essentially bounded, strongly measurable functions from $[0,T]$ into the set $B(Y, \mathscr{H})$ of bounded operators from Y into $\mathscr{H}$.*

Under the assumptions (A1)-(A4), for an initial datum $U_0 \in \mathscr{H}$, there exists a unique mild solution $U \in C([0,T], \mathscr{H})$ of system (12).
Moreover, if the initial datum $U_0 \in \mathscr{D}(\mathscr{A}(t))$, then there exists a unique strong solution

$$U \in C([0,T], \mathscr{D}(\mathscr{A}(t))) \cap C^1([0,T], \mathscr{H})$$

of system (12).

Let now check the above assumptions for the system (12). We will verify that the operator $\mathscr{A}(t)$ generates a C_0-semigroup in $\mathscr{H}$ and the system (12) has a unique solution (and then system (1)) by using the variable norm technique of Kato [23].

In this sense, we introduce the following time-independent inner product on $\mathscr{H}$.
For $\Phi = (\phi_1, \phi_2, \phi_3, \phi_4)^T$ and $W = (w_1, w_2, w_3, w_4)^T$ in $\mathscr{H}$, we consider

$$\begin{aligned} \langle \Phi, W \rangle_t \quad := \quad & \langle \phi_1, w_1 \rangle_{D(A^{\frac{1}{2}})} - h_0 \langle \phi_1, w_1 \rangle_{D(B^{\frac{1}{2}})} + \langle \phi_2, w_2 \rangle + \langle \phi_3, w_3 \rangle_{L^2_h(\mathbb{R}_+, D(B^{\frac{1}{2}}))} \\ & + \xi \tau(t) \langle \phi_4, w_4 \rangle_{L^2(0,1;H)}, \end{aligned}$$

where ξ be a positive constant such that

$$\frac{|\mu_2|}{1-d} \leq \xi \leq 2\mu_1 - |\mu_2|. \tag{14}$$

with associated norm denoted by $\|.\|_t$. Note that it exists according to (10) and this new inner product is equivalent to the usual inner product on $\mathscr{H}$.

The well-posedness of solution of (12) is insured by the following theorem.

Theorem 2 *Under the assumptions (A1)-(A4), for an initial datum $U_0 \in \mathscr{H}$. There exists a unique mild solution $U \in C([0,+\infty), \mathscr{H})$ to system (12). Moreover, if $U_0 \in D(\mathscr{A}(t))$, then the solution of (12) satisfies (classical solution)*

$$U \in C(\mathbb{R}_+, \mathscr{D}(\mathscr{A}(t))) \cap C^1(\mathbb{R}_+, \mathscr{H}).$$

3. STABILITY RESULTS

In this section, we present some technical lemmas needed to prove our main results. Then, we show the stability results of solution by constructing a suitable Lyaponov functional with which we can show the desired results. The stability results hold under the following additional assumptions :

(A5) There exist a positive constant a_1 satisfying

$$\left\|A^{\frac{1}{2}}u\right\|^2 \leq a_1 \left\|B^{\frac{1}{2}}u\right\|^2, \quad \forall u \in D(A^{\frac{1}{2}}). \tag{15}$$

(A6) There exists a C^1 function $G:(0,+\infty)\to(0,+\infty)$ which is linear or it is strictly increasing and strictly convex C^2 function on $(0,r]$, $r \leq h(0)$, $G(0)=G'(0)=0$,
with $\lim_{t\to+\infty} G'(t)=+\infty$ such that

$$h'(t) \leq -\zeta(t)G(h(t)), \quad \forall t \geq 0, \tag{16}$$

where $\zeta:\mathbb{R}_+\to\mathbb{R}_+$ is a nonincreasing differentiable function.

Remark 2 *If G is a strictly increasing and strictly convex C^2 function on $(0,r]$, with $G(0)=G'(0)=0$, then G has an extension $\overline{G}$ which is a strictly increasing and strictly increasing C^2 function on $(0,+\infty)$. Moreover, we can define $\overline{G}$ by*

$$\overline{G}(t) = \frac{c}{2}t^2 + (b-cr)t + \left(a + \frac{c}{2}r^2 - br\right), \quad \text{for} \quad t > r, \tag{17}$$

where $a=G(r)$, $b=G'(r)$ and $c=G''(r)$.

Now, we define the energy functional E by

$$\begin{aligned} E(t) &= \frac{1}{2}\left(\left\|A^{\frac{1}{2}}u\right\|^2 - h_0\left\|B^{\frac{1}{2}}u\right\|^2 + \|u_t\|^2 + \int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds\right) \\ &\quad + \frac{\xi\tau(t)}{2}\int_0^1 \|z(\rho,t)\|^2 d\rho, \qquad \forall t \in \mathbb{R}_+, \end{aligned} \tag{18}$$

where ξ satisfies (14).
Let establish some several Lemmas in order to prove the desired results.

Lemma 3 *Let U be solution of problem (6). Then, the energy functional defined by (18) satisfies*

$$E'(t) \leq \frac{1}{2}\int_0^{+\infty} h'(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds \leq 0, \quad \forall t \in \mathbb{R}_+. \tag{19}$$

Lemma 4 *Let U be the solution of (6). Then the functional*

$$I_1(t) = \langle u_t(t), u(t)\rangle, \tag{20}$$

satisfies, for $\delta_1>0$ and for all $t\geq 0$

$$\begin{aligned} I_1'(t) &\leq \left(1+\frac{\mu_1}{4\delta_1}\right)\|u_t\|^2 - \left(\frac{l}{2} - \frac{a}{b}\delta_1(\mu_1+|\mu_2|)\right)\left\|A^{\frac{1}{2}}u\right\|^2 + \frac{|\mu_2|}{4\delta_1}\|z(1,t)\|^2 \\ &\quad + \frac{aC_\alpha}{2l}\int_0^{+\infty} k(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds, \end{aligned} \tag{21}$$

for any $0<\alpha<1$, where

$$C_\alpha = \int_0^{+\infty} \frac{h^2(s)}{\alpha h(s) - h'(s)} ds \quad \text{and} \quad k(t) = \alpha h(t) - h'(t) \tag{22}$$

and $l = 1 - ah_0$.

Lemma 5 *Let U be solution of (6). Then the functional*

$$I_2(t) = -\left\langle u_t(t), \int_0^{+\infty} h(s)\eta^t(s)ds \right\rangle, \tag{23}$$

satisfies, for $\varepsilon > 0$

$$\begin{aligned} I_2'(t) \quad &\leq \quad (\varepsilon - h_0)\|u_t\|^2 + \varepsilon \left\| A^{\frac{1}{2}} u \right\|^2 + \mu_2 \left\langle z(1,t), \int_0^{+\infty} h(s)\eta^t(s)ds \right\rangle \\ &\quad + \frac{c(C_\alpha + 1)}{\varepsilon} \int_0^{+\infty} k(s) \left\| B^{\frac{1}{2}} \eta^t(s) \right\|^2 ds, \end{aligned} \tag{24}$$

where $c = \max\left\{ \frac{\int_0^{+\infty} k(s)ds}{b}, \varepsilon + \frac{a_1}{2} + \frac{\mu_1^2}{2b} + \frac{ah_0^2}{2} + \frac{\alpha^2}{b} \right\}$.

Lemma 6 *Let U be solution of (6). Then the functional*

$$I_3(t) = \int_0^1 e^{-2\tau(t)\rho} \|z(\rho,t)\|^2 ds, \tag{25}$$

satisfies,

$$I_3'(t) \leq -2I_3(t) - \frac{(1-d)e^{-2\tau_1}}{\tau_1} \|z(1,t)\|^2 + \frac{1}{\tau_0} \|u_t\|^2. \tag{26}$$

Now, we need to state and prove the following result.

Lemma 7 *There exists a positive constant C_1 such that*

$$\int_t^{+\infty} h(s) \left\| B^{\frac{1}{2}} \eta^t(s) \right\|^2 ds \leq C_1 q_0(t), \tag{27}$$

where $q_0(t) = \int_0^{+\infty} h(t+s) \left(1 + \left\| B^{\frac{1}{2}} u_0(s) \right\|^2 \right) ds$.

Lemma 8 *Let U be solution of (6). Then the functional*

$$I_4(t) = \int_0^t f(t-s) \left\| B^{\frac{1}{2}} u(s) \right\|^2 ds, \tag{28}$$

satisfies, for all $t \geq 0$,

$$I_4'(t) \leq 3(1-l) \left\| A^{\frac{1}{2}} u \right\|^2 - \frac{1}{2} \int_0^{+\infty} h(s) \left\| B^{\frac{1}{2}} \eta^t(s) \right\|^2 ds + \frac{C_1}{2} q_0(t), \quad \forall t \geq 0, \tag{29}$$

where $f(t) = \int_t^{+\infty} h(s)ds$.

Now, let us construct a Lyapunov functional L as follows

$$L(t) = ME(t) + N_1 I_1(t) + N_2 I_2(t) + N_3 I_3(t), \tag{30}$$

where M, N_1 and N_2 are positive constants.

Lemma 9 *Assume that (A1)-(A4) hold, there exist two positive constants c_1 and c_2 such that*

$$c_1 E(t) \leq L(t) \leq c_2 E(t). \tag{31}$$

Lemma 10 *The Lyapunov functional L defined in* (30) *satisfies*

$$L'(t) \leq -4(1-l)\left\|A^{\frac{1}{2}}u\right\|^2 - \|u_t\|^2 + \frac{1}{4}\int_0^{+\infty} h(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds, \quad \forall t \geq 0, \tag{32}$$

under a suitable choice of M, N_1, N_2 and N_3.

At this Position, we should introduce the following functions before going to present the theorem of the decay results :

$$G_1(t) = \int_t^1 \frac{ds}{G_2(s)}, \quad \text{with} \quad G_2(t) = tG'(r_1 t), \tag{33}$$

where $0 < r_1 < r$. We have, based on the properties of G, G_1 is strictly decreasing and convex function on $(0,1]$ and $\lim_{t\to 0} G_1(t) = +\infty$. In addition to that, let S be the class of functions $\chi : \mathbb{R}_+ \to \mathbb{R}_+^*$ satisfy, for fixed positive constants k_1 and k_2,

$$\chi \in C^1(\mathbb{R}_+), \quad \chi \leq 1, \quad \chi' \leq 0, \tag{34}$$

and

$$k_2 G\left(\frac{c}{p_1} q(t) q_0(t)\right) \leq k_1 \left[G_2\left(\frac{G_3(t)}{\chi(t)}\right) - \frac{G_2\left(G_3(t)\right)}{\chi(t)}\right], \tag{35}$$

where $p_1 > 0$, q is a function defined as follows

$$q(t) := \frac{p}{\left(1 + \int_0^t q_0(s)ds\right)}, \tag{36}$$

where p is a positive constant such that $p < 1$ and

$$G_3(t) = G_1^{-1}\left(k_1 \int_0^t \zeta(s) ds\right). \tag{37}$$

Remark 3 *The set S is not empty because it contains $\chi(s) = \varepsilon_0 G_3(s)$ for any $0 < \varepsilon_0 \leq 1$ small enough. Indeed, by* (33) *and* (37), (34) *is satisfied. Moreover, we have $q(t)q_0(t)$ is nonincreasing, $0 < G_3 \leq 1$, G and G' are increasing, then,* (35) *is satisfied if*

$$k_2 G\left(\frac{c}{p_1} p q_0(0)\right) \leq \frac{k_1}{\varepsilon_0}\left[G'\left(\frac{r_1}{\varepsilon_0}\right) - G'(r_1)\right],$$

which holds, for $0 < \varepsilon_0 \leq 1$ small enough, since $\lim_{t\to+\infty} G'(t) = +\infty$. But with the choice $\chi = \varepsilon_0 G_3$, (38) *does not lead to any stability estimate. So, the best possible decay rate for E given by* (38) *depends to the choice of χ.*

The stability results is ensuring by the following theorem.

Theorem 11 *Assume that (A1)-(A4) hold. Then, there exist a positive constant C such that, for any χ satisfying* (34) *and* (35)*, the solution of* (1) *satisfies, for all $t \geq 0$,*

$$E(t) \leq C \frac{G_3(t)}{\chi(t) q(t)}. \tag{38}$$

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4. REFERENCES

[1] F. Alabau-Boussouira, P. Cannarsa, A general method for proving sharp energy decay rates for memory-dissipative evolution equations, C R Math, 347(15-16), 867-872 (2009).

[2] A. M. Al-Mahdi, General stability result for a viscoelastic plate equation with past history and general kernel, J Math Anal Appl, 124216 (2020).

[3] K. Ammari, S. Nicaise and C. Pignotti, Feedback boundary stabilization of wave equations with interior delay, Syst Control Lett, 59, 623-628 (2010).

[4] V. Arnold, Mathematical methods of classical mechanics. Springer, New York 1989.

[5] A. Bàtkai, S. Piazzera, Semigroups for delay equations, Research Notes in Mathematics, 10. AK Peters, Ltd., Wellesley, MA, (2005).

[6] F. Belhannache, M. M. Algharabli and S.A. Messaoudi, Asymptotic Stability for a Viscoelastic Equation with Nonlinear Damping and Very General Type of Relaxation Functions, J Dyn Control Syst, 1-23 (2019).

[7] A. Benaissa, A. Benguessoum and S. A. Messaoudi, Global existence and energy decay of solutions to a viscoelastic wave equation with a delay term in the non-linear internal feedback, International Journal of Dynamical Systems and Differential Equations, 5(1), 1-26 (2014).

[8] Y. Boukhatem, B. Benabderrahmane, General decay for a viscoelastic equation of variable coefficients with a time-varying delay in the boundary feedback and acoustic boundary conditions, ACTA Math Sci, 37(5), 1453-1471 (2017).

[9] Y. Boukhatem, B. Benabderrahmane, General Decay for a Viscoelastic Equation of Variable Coefficients in the Presence of Past History with Delay Term in the Boundary Feedback and Acoustic Boundary Conditions, ACTA Appl Math, 154(1), 131-152 (2018).

[10] M.M. Cavalcanti, A.D. Cavalcanti, I. Lasiecka and X. Wang, Existence and sharp decay rate estimates for a von Karman system with long memory, NA : Real World Appl, 22, 289-306 (2015).

[11] H. Chellaoua, Y. Boukhatem, Stability result for an abstract delayed evolution equation with arbitrary decay of viscoelasticity, Proc Inter Math Sci, 7-25 (2020).

[12] H. Chellaoua, Y. Boukhatem, Stability results for second-order abstract viscoelastic equation in Hilbert spaces with time-varying delay, Z Angew Math Phys, 72(46), (2021) https ://doi.org/10.1007/s00033-021-01477-y

[13] H. Chellaoua, Y. Boukhatem, Optimal decay for second-order abstract viscoelastic equation in Hilbert spaces with infinite memory and time delay, Math Meth Appl sci, (2020) https ://doi.org/10.1002/mma.6917.

[14] H. Chellaoua, Y. Boukhatem and B. Feng, Optimal decay of an abstract nonlinear viscoelastic equation in Hilbert spaces with delay term in the nonlinear internal damping, Asymptotic Anal, (2021) https ://doi.org/10.3233/ASY-201664.

[15] CM. Dafermos, Asymptotic stability in viscoelasticity, Arch Ration Mech An, 37, 297-308 (1970).

[16] R. Datko, Two questions concerning the boundary control of certain elastic systems, J Differ Equations, 92(1), 27-44 (1991).

[17] M. Fabrizio, S. Polidoro, Asymptotic decay for some differential systems with fading memory, Appl Anal, 81(6), 1245-1264 (2002).

[18] B. Feng, Global Well-Posedness and Stability for a Viscoelastic Plate Equation with a Time Delay, Math Probl Engin, 2015 (2015).

[19] A. Guesmia, Well-posedness and exponential stability of an abstract evolution equation with infinite memory and time delay, IMA J Math Cont Infor, 30(4), 507-526 (2013).

[20] A. Guesmia, Some well-posedness and stability results for abstract hyperbolic equations with infinite memory and distributed time delay, Comm Pure Appl Anal, 14(2), 457-491 (2014).

[21] A. Guesmia, New general decay rates of solutions for two viscoelastic wave equations with infinite memory, Math Model Anal, 25(3), 351-373 (2020).

[22] J. H. Hassan, S. A. Messaoudi, General decay rate for a class of weakly dissipative second-order systems with memory, Math Meth Appl Sci, 42(8), 2842-2853 (2019).

[23] T. Kato, Linear and quasilinear equations of evolution of hyperbolic type, in Hyperbolicity : Lectures given at a Summer School of the Centro Internazionale Matematico Estivo (C.I.M.E.), Cortona (Arezzo), Italy. Vol.2, G. da Prato and G. Geymonat, eds : Liguori Editore, 125-191 (1977).

[24] T. Kato, Abstract Differential Equations and Nonlinear Mixed Problems, Lezioni Fermiane [Fermi Lectures]. Scuola Normale Superiore : Pisa ; 1985.

[25] M. Kirane, B. Said-Houari, Existence and asymptotic stability of a viscoelastic wave equation with a delay, Z Angew Math Phys, 62(6) 1065-1082 (2011).

[26] I. Lasiecka, S. A. Messaoudi and M. I. Mustafa, Note on intrinsic decay rates for abstract wave equations with memory, J Math Phys, 54(3), 031504 (2013).

[27] S. A. Messaoudi, General decay of solutions of a viscoelastic equation, J Math Anal Appl, 341(2), 1457-1467 (2008).

[28] S. A. Messaoudi, General stability in viscoelasticity, viscoelastic and viscoplastic materials, Mohamed El-Amin, IntechOpen 2016. https ://doi.org/10.5772/64217. Available from : https ://www.intechopen.com/books/viscoelastic-and-viscoplastic-materials/general-stability-in-viscoelasticity.

[29] J. E. Muñoz Rivera, A. Peres Salvatierra, Asymptotic behaviour of the energy in partially viscoelastic materials, Q Appl Math, 59(3), 557-578 (2001).

[30] J. E. Muñoz Rivera, M. G. Naso and F. M. Vegni, Asymptotic behavior of the energy for a class of weakly dissipative second-order systems with memory, J Math Anal Appl, 286(2), 692-704 (2003).

[31] M. I. Mustafa, On the control of the wave equation by memory-type boundary condition, Discrete Conti Dyn-A, 35(3), 1179 (2014).

[32] M. I. Mustafa, Memory-type plate system with nonlinear delay, Adv Pure Appl Math, 8(4), 227-240 (2017).

[33] M. I. Mustafa, Optimal decay rates for the abstract viscoelastic equation, J Evol Equ, 1-17 (2019).

[34] S. Nicaise, C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, SIAM J Control Optim, 45, 1561-1585 (2006).

[35] S. Nicaise, C. Pignotti, Stabilization of second-order evolution equations with time delay, Math Control Signal, 26(4), 563-588 (2014).

[36] S. Nicaise S, J. Valein and E. Fridman, Stability of the heat and of the wave equations with boundary time-varying delays, Disc Cont Dyn Syst, Series 5, 2(3), 559-581 (2009).

[37] V. Pata, Exponential stability in linear viscoelasticity, Q Appl Math, 64(3), 499-513 (2006).

[38] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer, New York 1983.

[39] T. J. Xiao, J. Liang, Coupled second order semilinear evolution equations indirectly damped via memory effects, J Differ Equations, 254(5), 2128-2157 (2013).