UNIQUENESS AND STABILITY OF PARAMETER IDENTIFICATION IN ELLIPTIC BOUNDARY VALUE PROBLEM

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ABSTRACT

This talk concerns uniqueness and stability of an inverse problem in PDE. Our problem consists in identifying two parameters $b(x)$ and $c(x)$ in the following boundary-value problem [3].

$$\begin{cases} Lu := -b(x)u''(x) + c(x)u'(x) = f(x), \\ u(0) = u(1) = 0, \end{cases} \tag{1}$$

from distributed observations u_1 (resp. u_2) associated to the source f_1 (resp. f_2). For one observation the solution is not unique but we prove, under some conditions, the uniqueness of the solution $p = (b,c)$ in the case of two observations. Furthermore, we derive a Hölder type stability result. The algorithm of reconstruction uses the least squares method. Finally, we present some numerical examples in the case of exact and noisy data to illustrate our method.

Key-words : Inverse problem, Least squares method. Levenberg-Marquardt algorithm.
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1. SETTING OF THE PROBLEM

The direct problem is to find the weak solution of the problem (1) : given $b(x)$, $c(x)$, and $f(x)$, find $u \in H_0^1(0,1)$, such that

$$\int_0^1 [u'(x)v(x)(b'(x)+c(x)) + u'(x)v'(x)b(x)]dx = \int_0^1 f(x)v(x)dx, \quad \forall v \in H_0^1(0,1). \tag{2}$$

The direct problem is well-posed under standard conditions $b \in C^1[0,1]$; $b(x) \geq b_0 > 0$, $c \in C[0,1]$; $c(x) \geq 0$ and $f \in L^2(0,1)$.

Our **inverse problem** is the parameter identification. Given (u,f) (for one observation) or (u_1,f_1) and (u_2,f_2) (for two observations), we reconstruct the pair of coefficients $p = (b,c)$.

It is well-known that such problem is typically ill-posed problem i.e : the solution can be non-unique and unstable.

2. STABILITY

In this section, we give a condition for which the solution of the inverse problem is unique. The equation $\Phi(p) = (u_1,u_2)$ has a unique solution $p = (b,c)$ if and only if the linear system $\{Lu_1 = f_1;\ Lu_2 = f_2\}$ has a unique solution with respect to (b,c).

2.1. Notations

First, we introduce some notations that will be used throughout this paper.

— In order to provide an abstract formulation of the inverse problem, we introduce the parameter space $M_{ad} = \{(b,c);\ b \in C^1[0,1],\ b(x) \geq b_0 > 0,\ c \in C[0,1],\ c(x) \geq 0\}$.

— The parameter is the pair $p_1 = (b_1,c_1),\ p_2 = (b_2,c_2) \in M_{ad}$.

— Consider $f_1, f_2 \in L^2(0,1)$.

— The mapping Φ that relates the parameter to the observation is defined by $\Phi(p_1) = (u_1,u_2),\ \Phi(p_2) = (v_1,v_2) \in Y = H^2(0,1) \times H_0^1(0,1)$, with
u_j (resp. v_j) solution of $-b_1 u_j'' + c_1 u_j' = f_j$ (resp. $-b_1 v_j'' + c_1 v_j' = f_j$), $j = 1,2$.

— We set $\Delta_1(x) = f_1 u_2' - f_2 u_1'$ and $\Delta_2(x) = f_1 v_2' - f_2 v_1'$.

The determinant of the previous system for $p_1 = (b_1,c_1)$ is $\dfrac{\Delta_1(x)}{b_1(x)}$.

Proposition 1 : Suppose that $\Delta(x) \equiv f_1(x)u_2'(x) - f_2(x)u_1'(x) \neq 0$, $\forall x \in (0,1)$. Then, the equation $\Phi(p) = (u_1,u_2)$ has a unique solution $p = (b_1,c_1)$.

Proposition 2 : Assume that : $\exists \alpha_1 > 0$, such that

$$|\Delta_1(x)| \geq \alpha_1 > 0, \quad \forall x \in [0,1]. \tag{3}$$

Then we have the following estimate

$$\|p_1 - p_2\|_{L^2} \leq C(p_1)\|\Phi(p_1) - \Phi(p_2)\|_{Y \times Y}. \tag{4}$$

Proposition 2 means that the operator $\Phi : p \mapsto u = (u_1,u_2)$ from $M_{ad} \subset L^2 \times L^2$ to $H^2 \times H^2$ is invertible in a neighbourhood of p_1, moreover Φ^{-1} is continuous (locally Lipschitz).

3. ALGORITHM

One of the most commonly used approaches for solving the inverse problem is by setting it as a least squares problem [1]. The solution $p = (b,c)$ realize the minimum of the functional

$$J(p) = \frac{1}{2}\left[\|\Phi_1(p) - d_1\|^2 + \|\Phi_2(p) - d_2\|^2\right], \quad \text{for } p \in M_{ad}, \tag{5}$$

where $\Phi_j(p) = u_j(p)$ is the operator solution and $(d_1,d_2) \in L^2 \times L^2$ is the measured data. To solve the least squares problem, we apply the **Levenberg-Marquardt** method [4] which consists in iterating the procedure :
p_0 : initial approximation, $p_{n+1} = p_n + h_n$, where h_n is the solution of the linearized equation

$$\Phi'^*(p_n)\Phi'(p_n)h_n + \alpha_n h_n = \Phi'^*(p_n)(d - \Phi(p_n)), \tag{6}$$

where $\Phi'(p)$ is the Fréchet-derivative of Φ [2] given by the theorem :
Theorem : Φ is Fréchet-differentiable. The partial derivatives are given by :

$$\begin{cases} \dfrac{\partial \Phi}{\partial b}(p;h) = \big(A(p)\big)^{-1}(hu''), \\[2ex] \dfrac{\partial \Phi}{\partial c}(p;k) = -\big(A(p)\big)^{-1}(ku'). \end{cases} \tag{7}$$

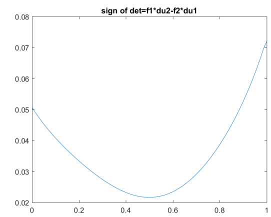
Where $A(p)$ is the differential operator defined by : $A(p) : \mathscr{D}(A) \longrightarrow H = L^2(0,1)$

$$\begin{cases} \mathscr{D}(A) = H^2(0,1) \cap H_0^1(0,1), \\ A(p)\varphi = L\varphi, \quad L\varphi = -b(x)\varphi'' + c(x)\varphi'. \end{cases} \tag{8}$$

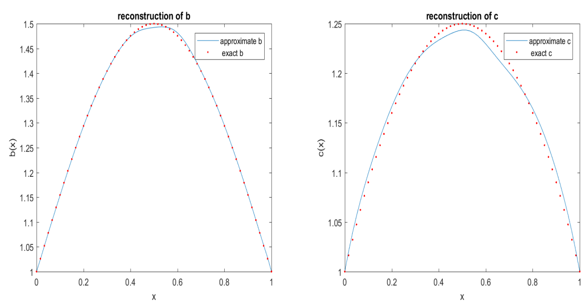
4. NUMERICAL EXAMPLES

As an example, we choose the coefficients $b(x) = 1 + 0.5\sin(\pi x)$ and $c(x) = 1 + x - x^2$.
Example 1 : $f_1(x) = \cos(\pi x)$ and $f_2(x) = x - x^2$. The solution u_1 and u_2 are computed with Finite Element Method.

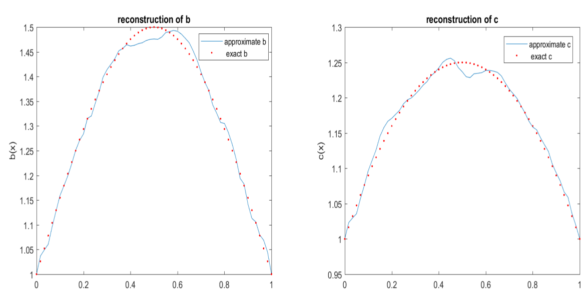
The following figure shows the variation of $\Delta(x)$



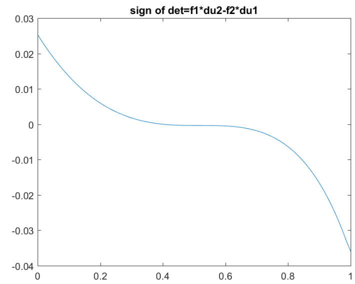
The following figures show reconstructions without noise



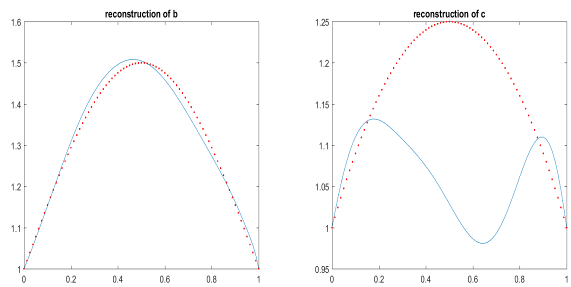
The following figures show reconstructions with noise-level $\delta = 10^{-5}$



Example 2 : $f_1(x) = \begin{cases} x & \text{if} \quad x \leq 0.5 \\ 0.5 & \text{if} \quad x > 0.5 \end{cases}$, $f_2(x) = x - x^2$.
The following figure shows the variation of $\Delta(x)$



The following figures show reconstructions without noise



4.1. Commentaries

- In example 1, Δ_1 does not change the sign ($\Delta_1(x) \geq 0.02$) ; which confirm the numerical stability.
- In example 2, Δ_1 changes the sign, it vanishes at $x = 0.5$. For the parameter $c(x)$, the algorithm converges to another solution (lack of uniqueness). However the parameter b is relatively stable.

5. REFERENCES

[1] G. Chavent, Nonlinear Least Squares for Inverse problems. Series Scientific Computation, Springer, 2009.

[2] H. W. Engl, M. Hanke, and A. Neubauer, Regularization of Inverse Problems, Kluwer Academic Publishers, 1996.

[3] M. Kern, Problèmes inverses, École superieure d'ingénieurs Léonard de Vinci, (2002-2003).

[4] J.J. Moré, The Levenberg-Marquardt algorithm : implementation and theory, in : G.A. Watson (Ed.), Lecture Notes in Mathematics 630 : Numerical Analysis, Springer-Verlag, Berlin, 1978, pp. 105–116.