NONTRIVIAL SOLUTION FOR QUASILINEAR ELLIPTIC SYSTEMS IN DIVERGENCE FORM

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ABSTRACT

In this paper we study the existence of nontrivial weak solutions for quasilinear elliptic system in divergence form. By means of the copmactness method, with suitable assumptions on the nonlinearities, we obtain the existence of nontrivial weak solutions.

1. INTRODUCTION

In this paper, we aim to investigate the existence of non-trivial solutions for the following quasilinear elliptic system

$$\begin{cases} -\mathrm{div}\Big(\sigma_1(x)a_1(x,u,\nabla u)\Big) = f_1 \quad \text{in } \Omega, \\ -\mathrm{div}\Big(\sigma_2(x)a_2(x,v,\nabla v)\Big) = f_2 \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is a bounded open subset of $\mathbb{R}^N$, $N \geq 1$, with smooth boundary $\partial\Omega$, $f = (f_1, f_2)$ is function in $(L^2(\Omega))^2$ and $\sigma = (\sigma_1, \sigma_2)$ is function in $(L^\infty(\Omega))^2$ satisfy

$$\text{there exists } \sigma_0 > 0 \text{ such that } \sigma \geq \sigma_0 \text{ a.e.} \tag{2}$$

Throughout this paper, $a_i(x,s,p),\ i = 1,2.$, satisfies the following conditions :

- (A_1) (Continuity) $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function, i.e., for any $s \in \mathbb{R}$ and $p \in \mathbb{R}^N$, $a_i(x,s,p)$ are both measurable in x for all $(s,p) \in \mathbb{R} \times \mathbb{R}^N$ and continuous in (s,p) for $a.e. x \in \Omega$.
- (A_2) (Coercivity) There exists $\beta_i > 0$ such that $a_i(x,s,p)p \geq \beta_i |p|^2,\ \forall p \in \mathbb{R}^N$.
- (A_3) (Growth) There exists $C_i \in \mathbb{R}$ such that $|a_i(x,s,p)| \leq C_i(1 + |s| + |p|),\ \forall (s,p) \in \mathbb{R} \times \mathbb{R}^N$.
- (A_4) (Monotonicity) $\big(a_i(x,s,p) - a_i(x,s,\eta)\big) \geq 0\ \forall (p,\eta) \in (\mathbb{R}^N)^2$.

Quasilinear elliptic equations and systems have been studied by many mathematician : see for example the works of Alves et al [1], Carmona et al [3], Carvalho et al [4], Edmunds and Webb [6], Mihailescu and Radulescu [12], Vishik [13] and the references therein. In recent years, elliptic partial differential equations have received lots attention, because its important role is playing in the real world and many perfect techniques. Notice that problems with Leray-Lions operator are widely studied in the literature, we refer the interested readers to [2, 5, 7, 8, 9]. Now let us briefly comment some known results of them.

In [5], the authors obtained the existence and multiplicity results for equations involving more general elliptic operators in divergence form via the mountain pass theorem. In [9], Norbert Hungerbühler considered the following Dirichlet problem for the quasilinear elliptic system

$$\begin{cases} -\mathrm{div}\sigma(x,u(x),Du(x)) = f, \quad x \in \Omega, \\ u(x) = 0, \quad x \in \partial\Omega, \end{cases} \tag{3}$$

for a function $u : \Omega \to \mathbb{R}^m$, where Ω is a bounded open domain in $\mathbb{R}^n$. Under classical regularity, they proved the existence of a weak solution for system (3) by Young measures. The corresponding results were further extended by Fu and Yang [7] to the case that σ satisfies variable growth conditions in system (3).

The aim of this work is to investigate the existence of nontrivial solutions to the quasilinear elliptic system (1). This existence is obtained by using the compactness method [11] and the monotonicity arguments.

The rest of this work is organized as follows. In the section 2, we present the main results. Then, by virtue of topological degree, we study the existence of nontrivial solutions for the broblem (1) in the section 3. Finally, we give a discussion about our research results in the last section.

2. MAIN RESULT

In the present section, we discuss the notions of weak solutions for problem (1) and the main result. First, let

$$W = H_0^1(\Omega) \times H_0^1(\Omega),$$

which is a Banach space endowed with the norm

$$\|(u,v)\|_W^2 = \|u\|_{H_0^1(\Omega)}^2 + \|v\|_{H_0^1(\Omega)}^2,$$

and let $\widetilde{W} = L^2(\Omega) \times L^2(\Omega)$, and $V = L^\infty(\Omega) \times L^\infty(\Omega)$. In the sequel, $\|\cdot\|_{H_0^1(\Omega)}$, $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{L^\infty(\Omega)}$ will denote the usual norms of $H_0^1(\Omega)$, $L^2(\Omega)$ and $L^\infty(\Omega)$, respectively.

We give now the definition of a weak solution for problem (1).

Definition 1 *We say that $(u,v) \in W$ is a weak solution for the system (1) if for any $(\phi,\varphi) \in W$ we have*

$$\begin{aligned} &\int_\Omega \sigma_1(x) a_1(x,u,\nabla u) \nabla\phi dx + \int_\Omega \sigma_2(x) a_2(x,v,\nabla v) \nabla\varphi dx \\ &= \int_\Omega f_1 \phi dx + \int_\Omega f_2 \varphi dx \end{aligned} \tag{4}$$

Now, we state our main results.

Theorem 1 *Assume that $(A_1) - (A_4)$ are satisfied. Then problem (1) has at least one solution.*

3. PROOF OF MAIN THEOREM

The main tool that we will use to prove the existence of nontrivial weak solutions of the problem (1) is the compactness method.

Let E be a finite-dimensional subspace of W endowed with the W-norm, and E^* its dual. We define the mappings : $T : E \times [0,1] \to E^*$ by

$$\begin{aligned}
&\langle T(u,v,\lambda),(\phi,\varphi)\rangle_{\mathrm{W}} \\
&= \int_\Omega \sigma_1(x)a_1(x,\lambda u,\lambda\nabla u)\nabla\phi\,\mathrm{d}x + \int_\Omega \sigma_2(x)a_2(x,\lambda v,\lambda\nabla v)\nabla\varphi\,\mathrm{d}x \\
&- \int_\Omega f_1(x)\phi\,\mathrm{d}x - \int_\Omega f_2(x)\varphi\,\mathrm{d}x,
\end{aligned} \tag{5}$$

for all $(\phi,\varphi) \in \mathrm{E}$, T is well defined. We shall prove the theorem in several steps.

Step 1 : A priori bounds. Using (2), (A_2) and Cauchy-schwartz inequalitie, we obtain a priori bounds of (u,v).

Step 2 : T is bounded. In this step, by using the assumption A_3 and the Cauchy-schwartz inequalitie, one has

$$T\left(\bar{B}^{\mathrm{E}}(R)\times[0,1]\right) \subset \bar{B}^{\mathrm{E}^*}\left(\widetilde{R}\right). \tag{6}$$

Step 3 : T is continuous. In this step we use $(A_1),(A_3)$, (6) and the dominated convergence theorem to prove that the mappings T is continuous.

Step 4 : Passing to the limit. To prove the passage to the limit, we use the assumption (A_4), the trick of Minty [10] and the following lemma :

Lemma 2 *If $a \in C(\mathbb{R}^N,\mathbb{R}^N)$, $a(x,s,p) \leq (1+|s|+|p|)$ for all $s \in \mathbb{R}$, $p \in \mathbb{R}^N$ and if $u_n \to u$ in $H_0^1(\Omega)$ then $a(x,u_n,\nabla u_n) \to a(x,u,\nabla u)$ in $L^2(\Omega)$.*

Lemma (2) is proved by the dominated convergence theorem of Lebesgue.

4. CONCLUSIONS

We considered the existence of nontrivial weak solutions for a quasilinear elliptic system in divergence form. By using the compactness method and the monotonicity arguments, we schowed that system (1) has at least one solutions when the coefficients a_1 and a_2 satisfying the classical Leray-Lions conditions.

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