EXISTENCE AND UNIQUENESS OF SOLUTION TO $G-$NEUTRAL STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT

In this research, we prove under suitable assumptions the existence and uniqueness of solution to neutral stochastic differential equations driven by $G-$Brownian motion. Our approach is based on the Caratheodory approximation schemes.

1. INTRODUCTION

Peng in 2006 developed the so-called $G-$expectation [6], which is a sub$-$linear expectation, generated by the $G-$heat equation, where G is an infinitesimal generator ; thereby, a stochastic process called $G-$Brownian motion, which plays a similar role to the classical Wiener process has been constructed. after that Peng set up the corresponding stochastic calculus of Itô type see [7]. Since then, stochastic differential equations driven by $G-$Brownian motion $(G-\text{SDEs shortly})$, have been introduced and attracted many author's attention, (see [1, 2, 5, 8] and references therein).

Following the way, we have proved under suitable assumptions the existence and uniqueness of the solution in $L_G^2(\Omega)$, to neutral stochastic differential equations driven by $G-$Brownian motion ($G-$NSDEs shortly) by using Caratheodory approximation schemes. Neutral stochastic differential equation introduced for the first time by Kolmanovskii and Myshkis [3, 4] in the theory of aeroelasticity and chemical engineering systems. Since then, the theory of NSDEs has attracted more attention and started to appear intensively in physics, chemical engineering, medicine, ecological sciences and other domains.

2. PROBLEM FORMULATION AND ASSUMPTIONS

Let r be a positive fixed time ; the canonical process $\{B_t, t \geq 0\}$ is a $d-$dimensional $G-$Brownian motion ; $BC = BC([-r,0];\mathbb{R}^n)$ is the space of $\mathbb{R}^n$ valued function Φ continuous, bounded and defined on $[-r,0]$ equipped with norm

$$\|\Phi\| = \sup_{-r\leq\theta\leq 0} |\Phi(\theta)|,$$

Considering the following $n-$dimensional $G-$NSDE

$$d[X(t) - Q(X_t)] = F(t, X_t)\,dt + G(t, X_t)\,d\langle B, B\rangle_t + H(t, X_t)\,dB_t \tag{1}$$

where $X_t : [-r,0] \to \mathbb{R}^n$, $X_t(\theta) = X(t+\theta)$, is the past history of the state and belongs to BC ; the functionals Q, F, G and H are defined as follows

$$Q : BC([-r,0];\mathbb{R}^n) \longmapsto \mathbb{R}^n \quad ; \quad F : [0,T] \times BC([-r,0];\mathbb{R}^n) \longmapsto \mathbb{R}^n,$$
$$G, H : [0,T] \times BC([-r,0];\mathbb{R}^n) \longmapsto \mathbb{R}^{n\times d},$$

all the coefficients Q,F,G and $H \in M_G^2([-r,T];\mathbb{R}^n)$ and Borel measurable. By the $G-$It$\widehat{o}$ integral the equation (1) can be written in the following equivalent form

$$X(t) = Q(X_t) + X(0) - Q(X_0) + \int_0^t F(s,X_s)\,ds + \int_0^t G(s,X_s)\,d\langle B,B\rangle_s \tag{2}$$
$$+\int_0^t H(s,X_s)\,dB_s, \quad \text{for all } 0 \le t \le T:$$

with the initial condition

$$X_0 = x = \{x(\theta) : -r \le \theta \le 0\} \in BC, \tag{3}$$

where $\mathbb{E}\|x\|^2 < \infty$, i.e., $x \in M_G^2([-r,0];\mathbb{R}^n)$.

2.1. Caratheodory approximations

For all $n \ge \frac{2}{r}$, we define $X^n(t)$ on $[-r,T]$ as follows

$$X^n(\theta) = x(\theta), \quad \theta \in [-r,0];$$

and

$$X^n(t) = Q(X_t^n) + x(0) - Q(x) + \int_0^t F(s,X^n_{s-\frac{1}{n}})ds + \tag{4}$$
$$\int_0^t G(s,X^n_{s-\frac{1}{n}})d\langle B,B\rangle_s + \int_0^t H(s,X^n_{s-\frac{1}{n}})dB_s, \quad \text{For } t \in [0,T]$$

We mention that $X^n(t)$ is calculated step by step on the intervals $\left[0,\frac{1}{n}\right), \left[\frac{1}{n},\frac{2}{n}\right), \ldots,$. For example, if $t \in \left[0,\frac{1}{n}\right]$ then $-\frac{1}{n} \le t - \frac{1}{n} \le 0$. Thus $X_t^n = x$, Also for $0 \le s \le \frac{1}{n}$ we have $-\frac{1}{n} \le s - \frac{1}{n} \le 0$ and $X^n_{s-\frac{1}{n}} = x$, consequently

$$X^n(t) = x(0) + \int_0^t F(s,x)ds + \int_0^t G(s,x)d\langle B,B\rangle_s + \int_0^t H(s,x)dB_s.$$

We stand the following assumptions

$(\mathbf{A_1})$: For all $\varphi,\psi \in BC([-r,0];\mathbb{R}^n)$, there exists a constant $C > 0$ such that

$$|F(s,\varphi) - F(s,\psi)|^2 \vee |G(s,\varphi) - G(s,\psi)|^2 \vee |H(s,\varphi) - H(s,\psi)|^2 \le C\|\varphi(0) - \psi(0)\|^2.$$

$(\mathbf{A_2})$: For all $\varphi \in BC([-r,0];\mathbb{R}^n)$,

$$|F(s,\varphi)|^2 \vee |G(s,\varphi)|^2 \vee |H(s,\varphi)|^2 \le \rho\left(1 + \|\varphi\|^2\right),$$

where $\rho(.) : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-decreasing and concave function, $\rho(0) = 0$ and $\rho(x) > 0$, for $x > 0$ with $\int_{0+} \frac{dx}{\rho(x)} = \infty$.

$(\mathbf{A_3})$: Suppose that $Q(0) = 0$, there exists a constant $\kappa \in]0,1[$ such that

$$|Q(\varphi) - Q(\psi)| \le \kappa\|\varphi(0) - \psi(0)\|,$$

For all $\varphi,\psi \in BC([-r,0];\mathbb{R}^n)$. This implies

$$|Q(\varphi)| \le \kappa\|\varphi\|.$$

3. EXISTENCE AND UNIQUENESS THEOREM

Theorem 1 *There exists a unique solution in $L_G^2(\Omega)$ of the equation* (2) *with the initial condition* (3).

In order to prove the above theorem, we need the following lemmas.

Lemma 2 *For all $t \in [-r,T]$ and $n \geq \frac{2}{r}$,*

$$\widehat{\mathbb{E}}\left(|X^n(t)|^2\right) \leq L_1$$

where L_1 is positive constant.

Lemma 3 *There exists positive constant L_2 such that*

$$\widehat{\mathbb{E}}|X^n(t) - X^n(s)|^2 \leq L_2|t-s|,$$

for all $0 \leq s < t \leq T$ and $n \geq \frac{2}{r}$.

Proof of theorem 4.1. The idea of the proof is firstly to show that $\{X^n(t)\}_{n \geq \frac{2}{r}}$ is a Cauchy sequence in $L_G^2(\Omega)$. Then, we prove that its limit is the unique solution of the equation (2). Let $\frac{2}{r} \leq n \leq m$, $\kappa \in]0,1[$, we have

$$\widehat{\mathbb{E}}\left(\sup_{0\leq s\leq t}|X^m(s) - X^n(s)|^2\right) \leq \frac{1}{(1-\kappa)^2}\widehat{\mathbb{E}}\left|\sup_{0\leq s\leq t}\Lambda_n(s)\right|^2$$

where

$$\begin{aligned}\widehat{\mathbb{E}}\left|\sup_{0\leq s\leq t}\Lambda_n(s)\right|^2 \leq\; & 3\widehat{\mathbb{E}}\left(\sup_{0\leq v\leq t}\left|\int_0^v F(s,X^m_{s-\frac{1}{m}}) - F(s,X^n_{s-\frac{1}{n}})ds\right|^2\right) \\ & +3\widehat{\mathbb{E}}\left(\sup_{0\leq v\leq t}\left|\int_0^v G(s,X^m_{s-\frac{1}{m}}) - G(s,X^n_{s-\frac{1}{n}})d\langle B,B\rangle_s\right|^2\right) \\ & +3\widehat{\mathbb{E}}\left(\left|\sup_{0\leq v\leq t}\int_0^v H(s,X^m_{s-\frac{1}{m}}) - H(s,X^n_{s-\frac{1}{n}})dB_s\right|^2\right).\end{aligned}$$

By Holder's inequality, **BDG**−inequalities and assumption (A_1), we obtain

$$\widehat{\mathbb{E}}\left(\sup_{0\leq s\leq t}|X^m(s) - X^n(s)|^2\right) \leq \frac{3C(K_1+TK_2+T)}{(1-\kappa)^2}\int_0^t \widehat{\mathbb{E}}\left|X^m\left(s-\frac{1}{m}\right) - X^n\left(s-\frac{1}{n}\right)\right|^2 ds \quad (5)$$

where K_1 and K_2 being positive constants. We use lemma 3, to get

$$\widehat{\mathbb{E}}\left|X^m\left(s-\frac{1}{m}\right) - X^n\left(s-\frac{1}{n}\right)\right|^2 \leq 2L_2\left|\frac{1}{n}-\frac{1}{m}\right| + 2\widehat{\mathbb{E}}\left(\sup_{0\leq v\leq s}|X^m(v) - X^n(v)|^2\right).$$

therefore, we substitute the above inequality in (5), then by Gronwall's lemma we deduce that

$$\widehat{\mathbb{E}}\left(\sup_{0\leq s\leq t}\left|X^m(s)-X^n(s)\right|^2\right)\leq L_3\left|\frac{1}{n}-\frac{1}{m}\right|e^{\frac{6C\left(K_1+TK_2+T\right)}{(1-\kappa)^2}T}$$

where $L_3=\frac{6TL_2C(K_1+TK_2+T)}{(1-\kappa)^2}$. It turns out that

$$\widehat{\mathbb{E}}\left(\sup_{0\leq s\leq t}\left|X^m(s)-X^n(s)\right|^2\right)\to 0,\ \text{when } n,m\to\infty.$$

Which shows that $\{X^n(t)\}_{n\geq\frac{2}{r}}$ is Cauchy sequence in $L_G^2(\Omega)$ and converges to some limit $X(t)$. It remains to prove that $X^n(t)\to X(t)$, when $n\to\infty$. From the equations (2) and (4), similarly to the above steps, one can show that

$$\widehat{\mathbb{E}}\left(\sup_{0\leq s\leq t}\left|X^n(s)-X(s)\right|^2\right)\leq\frac{TL_2L_4}{n}e^{L_4T}.$$

where $L_4=\frac{6C(K_1+TK_2+T)}{(1-\kappa)^2}$. Consequently

$$\widehat{\mathbb{E}}\left(\sup_{0\leq s\leq t}\left|X^n(s)-X(s)\right|^2\right)\to 0,\ \text{when } n\to\infty.$$

This means that $X^n(t)$ converges to $X(t)$ in $L_G^2(\Omega)$. The proof is complete. ■

4. CONCLUSIONS

In this study, we have proved the existence-uniqueness of the solution in $L_G^2(\Omega)$ to NSDEs driven by $G-$Brownian motion. Our approach is the Caratheodory approximation schemes. We have shown that the approximations of the solution construct a Cauchy sequence in the Banach space $L_G^2(\Omega)$ Thus converges to a unique limit which is the exact solution of the equation.

5. REFERENCES

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