SPDEs with space interactions and application to population modelling

Nacira Agram[1], Astrid Hilbert[1], Khouloud Makhlouf [2] and Bernt Øksendal[3]

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Abstract

We consider optimal control of a new type of non-local stochastic partial differential equations (SPDEs). The SPDEs have *space interactions*, in the sense that the dynamics of the system at time t and position in space x also depend on the space-mean of values at neighbouring points. This is a model with many applications, e.g. to population growth studies and epidemiology. We prove the existence and uniqueness of solutions of a class of SPDEs with space interactions, and we show that, under some conditions, the solutions are positive for all times if the initial values are. Sufficient and necessary maximum principles for the optimal control of such systems are derived. Finally, we apply the results to study an optimal vaccine strategy problem for an epidemic by modelling the population density as a space-mean stochastic reaction-diffusion equation.

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1 Introduction

The purpose of this paper is to introduce a new type generalised stochastic heat equation with *space interactions* as a model for population growth. By space interactions we mean that the dynamics of the population density $Y(t,x)$ at a time t and a point x depends not

[1]Department of Mathematics, Linnaeus University, Växjö, Sweden.
Email: nacira.agram@lnu.se, astrid.hilbert@lnu.se.
[2]Department of Mathematics, University of Biskra, Algeria. Email: khouloud.makhlouf@univ-biskra.dz.
[3]Department of Mathematics, University of Oslo, Norway. Email: oksendal@math.uio.no.

only on its value and derivatives at x, but also on its values in a neighbourhood of x. For example, define G to be a space-averaging operator of the form

$$G(x, \varphi) = \frac{1}{V(K_r)} \int_{K_r} \varphi(x+y) dy; \quad \varphi \in L^2(\mathbb{R}^n), \tag{1.1}$$

where $V(\cdot)$ denotes Lebesgue volume and

$$K_r = \{y \in \mathbb{R}^n; |y| < r\}$$

is the ball of radius $r > 0$ in $\mathbb{R}^n$ centred at 0. Then

$$\overline{Y}_G(t, x) := G(x, Y(t, \cdot))$$

is the average value of $Y(t, x + \cdot)$ in the ball K_r.
More generally, if we are given a nonnegative measure (weight) $\rho(dy)$ of total mass 1, then the ρ-weighted average of Y at x is defined by

$$\overline{Y}_\rho(t, x) := \int_D Y(t, x+y) \rho(dy).$$

We believe that by allowing interactions between populations at different locations, we get a better model for population growth, including the modelling of epidemics. For example, we know that COVID-19 is spreading by close contact in space.

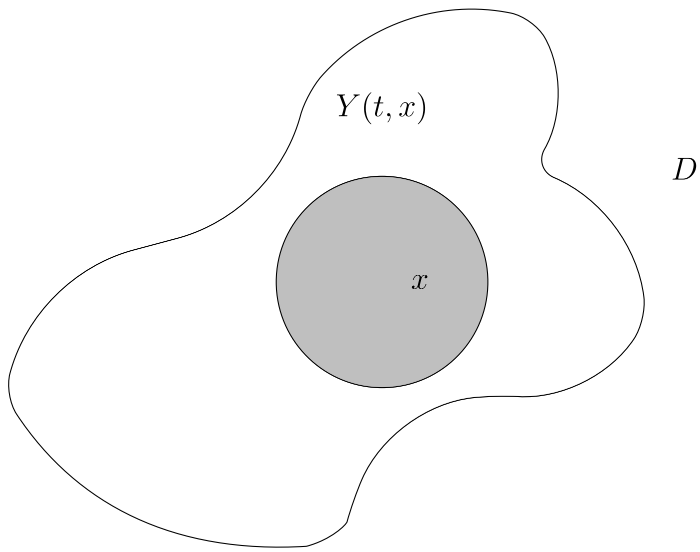
We illustrate the above by the following population growth model:

Example 1.1 *With G as in (1.1), suppose the density $Y(t, x)$ of a population at the time t and the point x satisfies the following space-interaction version of a reaction-diffusion equation:*

$$\begin{cases} dY(t,x) &= \Big(\frac{1}{2}\Delta Y(t,x) + \alpha \overline{Y}(t,x) - u(t,x)Y(t,x)\Big)dt + \beta Y(t,x) dB(t), \\ Y(0,x) &= \xi(x); \quad x \in D, \\ Y(t,x) &= \eta(t,x); \quad (t,x) \in (0,T) \times \partial D, \end{cases} \tag{1.2}$$

where α is a constant, ξ, η are given bounded functions and $\overline{Y}(t, x) = G(x, Y(t, \cdot))$.

Here $u(t,x)$ is our control process, e.g. representing our harvesting or vaccine effort.



Then (1.2) is a natural model for population growth in an environment with space interactions.

If $u(t,x)$ represents a vaccination effort rate at (t,x), we define the total expected utility $J_0(u)$ of the harvesting by an expression of the form

$$J_0(u) = \mathbb{E}\Big[\int_D \int_0^T U_1(u(t,x))dtdx + \int_D U_2(Y(T,x))dx\Big],$$

where U_1 and U_2 are given cost functions. The problem to find the optimal vaccination rate u^* is the following:

Problem 1.1 *Find $u^* \in \mathcal{U}$ such that*

$$J_0(u^*) = \inf_{u \in \mathcal{U}} J_0(u),$$

where $\mathcal{U}$ is a given family of admissible controls.

We will return to the example above after first discussing more general stochastic optimal control models with a system whose state $Y(t,x)$ at time t and at the point x satisfies an SPDE with a non-local space-interaction dynamics of the following type:

$$\begin{cases} dY(t,x) &= A_x Y(t,x)dt + b(t,x,Y(t,x),Y(t,\cdot),u(t,x))dt \\ &\quad +\sigma(t,x,Y(t,x),Y(t,\cdot),u(t,x))dB(t), \\ Y(0,x) &= \xi(x); \quad x \in D, \\ Y(t,x) &= \eta(t,x); \quad (t,x) \in (0,T) \times \partial D. \end{cases} \tag{1.3}$$

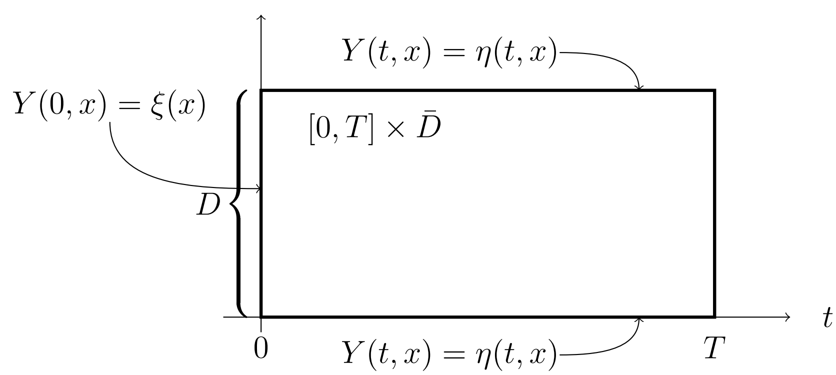
Here $dY(t,x)$ denotes the differential with respect to t while A_x is the second order partial differential operator acting on x of the form

$$A_x\phi(x) = \sum_{i,j=1}^{n} \alpha_{ij}(x) \frac{\partial^2 \phi}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \beta_i(x) \frac{\partial \phi}{\partial x_i}; \quad \phi \in \mathcal{C}_0^2(\mathbb{R}^n). \tag{1.4}$$

Precise conditions on the coefficients will be given in the beginning of Section 4.2.

The domain D is an open set in $\mathbb{R}^n$ with a Lipschitz boundary ∂D and closure $\overline{D}$. We extend $Y(t,x)$ to be a function on all of $[0,T] \times \mathbb{R}^n$ by setting

$$Y(t,x) = 0 \text{ for } x \in \mathbb{R}^n \setminus \overline{D}.$$



There are two well-known approaches to solve stochastic control problems: The Bellman dynamic programming method and the Pontryagin maximum principle. Because of the space-mean dependence in our model, the system is not Markovian, and it is not clear how to apply a dynamic programming approach. In stead we will use a stochastic version of the Pontryagin maximum principle, which involves a coupled system of a forward/backward SPDEs,

In the classical case when there is no interaction from neighbouring places, stochastic control of SPDEs has been studied widely in the literature, for example, we refer to Bensoussan [3], [4], [5], [6], Hu & Peng [18], Zhou [33], Øksendal [22], Fuhrman et a [12] and Øksendal *et al* [23], [24], [25] and the references therein.

In the case of a control problem for an SPDE with space-interaction dynamics we derive an adjoint process, which is a backward SPDE with space-interaction dependence. For related singular stochastic control with space-interaction, we refer to Agram *et al* [1].

More details about the theory of SPDE, we refer for example to Gawarecki & Mandrekar [14], Da Prato & Zabczyk [28], Pardoux [26], [27], Hairer [19], Prévôt & Roeckner [29] and to Roeckner & Zhang [30].

Here is a summary of the content of this paper:

- In Section 2 we prove the existence and uniqueness of the solution of a class of space-interaction SPDEs, including the application studied in Section 5, and we give an iterative procedure for finding the solution (Theorem 2.1). This result is new.

- In Section 3 we use white noise theory to prove a useful positivity theorem for a class of SPDEs with space interactions (Theorem 3.1), and we prove that the solution is positive if the initial values are (Theorem 3.2).These results are also new and of independent interest.

- Finally, as an illustration of our results, in Section 5 we study an example about optimal vaccination strategy for an epidemics modelled as an SPDE with space-interactions.

2 Solutions of SPDEs with space interactions, and positivity

In this section we prove an existence and uniqueness result for solutions of SPDEs with space interactions. We are not aiming at proving this for the most general SPDE of this type, but we settle for a class of SPDEs which includes the application in Section 5. Thus, for simplicity we consider only the case when $A_x = L$ given by

$$L = \frac{1}{2}\Delta := \frac{1}{2}\sum_{k=1}^{k=n} \frac{\partial^2}{\partial x_k^2}, \text{ and } D = \mathbb{R}^n,$$

but it is clear that our method can also be applied to more general situations.
Fix $t > 0$, and let $k \in \mathbb{N}_0 = \{0, 1, 2, \ldots, \ldots\}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m) \in \mathbb{N}_0^m; m = 1, 2, ...$
For functions $f \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ (the family of functions in $\mathcal{C}(\mathbb{R}^n)$ with compact support), we define the Sobolev norm

$$|f|_k = \sum_{|\alpha| \le k} \big(\int_{\mathbb{R}^n} |\partial^\alpha f(x)|^2 dx \big)^{\frac{1}{2}}; \alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{N}_0^n,$$

and we define the Sobolev space $\mathbb{H}^k$ to be the closure of $\mathcal{C}_0^\infty(\mathbb{R}^n)$ in this norm.
Note that $\mathbb{H}^k$ is a Hilbert space for all k.

Also, note that if $f \in \mathbb{H}^{k+2}$ then $Lf \in \mathbb{H}^k$, because

$$|Lf|_k = \sum_{|\alpha| \le k} \big(\int_{\mathbb{R}^n} |\partial^\alpha Lf(x)|^2 dx \big)^{\frac{1}{2}} \le \frac{1}{2} \sum_{|\alpha| \le k+2} \big(\int_{\mathbb{R}^n} |\partial^\alpha f(x)|^2 dx \big)^{\frac{1}{2}} = \frac{1}{2}|f|_{k+2}. \tag{2.1}$$

Let $\mathcal{Y}_k^{(t)}$ denote the family of adapted random fields $Y(s,x) = Y(s,x,\omega)$, such that $||Y||_k^{(t)} < \infty$ where

$$||Y||_{t,k} = \mathbb{E}\left[\sup_{s \le t} \left\{ |Y(s,.)|_k^2 \right\}\right]^{\frac{1}{2}}, \tag{2.2}$$

and let $\mathcal{Y}^{(t)}$ be the intersection of all the spaces $\mathcal{Y}_k^{(t)}; k \in \mathbb{N}_0$, with the norm

$$||Y||_t^2 := \sum_{k=1}^{\infty} 2^{-k} ||Y||_{t,k}^2. \tag{2.3}$$

In the following we let

$$\varphi \mapsto \overline{\varphi}(x)$$

be any averaging operator such that there exists a constant C_1 such that

$$|\overline{\varphi}|_k \leq C_1 |\varphi|_k \text{ for all } \varphi, k. \tag{2.4}$$

This holds, for example, if $\overline{\varphi}(x) = \int \varphi(x+y)\rho(dy)$ for some measure ρ of total mass 1.

We can now prove the following:

Theorem 2.1 *Let $\xi \in \mathcal{Y}^{(T)}$ be deterministic and let $h : [0,T] \mapsto \mathbb{R}$ be bounded and deterministic.*

(i) Then there exists a unique solution $Y(t,x) \in \mathcal{Y}^{(T)}$ of the following SPDE with space interactions:

$$\begin{aligned} Y(t,x) &= \xi(x) + \int_0^t LY(s,x)ds \\ &\quad + \int_0^t \overline{Y}(s,x)ds + \int_0^t h(s)Y(s,x)dB(s); \quad t \in [0,T]. \end{aligned}$$

(ii) Moreover, the solution $Y(t,x)$ can be found by iteration, as follows: Choose $Y_0 \in \mathcal{Y}^{(T)}$ arbitrary deterministic and define inductively Y_m to be the solution of

$$\begin{aligned} Y_m(t,x) &= \xi(x) + \int_0^t LY_m(s,x)ds + \int_0^t \overline{Y}_{m-1}(s,x)ds \\ &\quad + \int_0^t h(s)Y_m(s,x)dB(s); \quad t \in [0,T]; m = 1,2,.... \end{aligned} \tag{2.5}$$

Then

$$Y_m \to Y \text{ in } \mathcal{Y}^{(T)} \text{ when } m \to \infty.$$

3 The non-homogeneous stochastic heat equation and positivity

In this section we will prove positivity of the solutions $Y(t,x)$ of SPDEs of the form

$$\begin{cases} dY(t,x) &= LY dt + K(t,x)\,dt + h(t)\,Y(t)\,dB(t), \\ Y(0,x) &= \xi(x); \quad x \in \mathbb{R}^n, \end{cases}$$

where the function $\xi \in \mathcal{Y}^{(T)}$ is deterministic and positive, $h : [0,T] \mapsto \mathbb{R}$ is bounded and deterministic and $K(t,x) = K(t,x,\omega) : [0,T] \times \mathbb{R}^n \times \Omega \mapsto \mathbb{R}$ is a given positive random field.

To motivate our method, we first recall the following basic results about the classical heat equation:
Let $L = \frac{1}{2}\triangle$ and consider the equation

$$\begin{cases} dY(t,x) &= LY\,dt + K(t,x)\,dt, \\ Y(0,x) &= \xi(x); \quad x \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

where $\xi \in \mathcal{Y}^{(T)}$ and $K \in L^2([0,T] \times \mathbb{R}^n)$ are given deterministic functions. Define the operator $P_t : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ by

$$P_t f(x) = \int_{\mathbb{R}^n} (2\pi t)^{-\frac{n}{2}} f(y) \exp\left(-\frac{|x-y|^2}{2t}\right) dy, \tag{3.2}$$

then

$$\frac{d}{dt} P_t f = L(P_t f),$$

and if we define

$$Y(t,x) = P_t \xi(x) + \int_0^t P_{t-s}(K(s,.))(x)\,ds,$$

we get

$$\begin{aligned} \frac{d}{dt} Y(t,x) &= L(P_t \xi)(x) + P_0(K(t,.))(x) + \int_0^t L(P_{t-s}(K(s,.)))(x)\,ds \\ &= LY(t,x) + K(t,x). \end{aligned}$$

Hence

$$Y(t,x) \text{ solves the heat equation (3.1).}$$

Next, consider the case

$$dY(t,x) = LY\,dt + K(t,x)\,dt + \theta(t)\,Y(t,x)\,dt.$$

Multiply the equation by

$$Z(t) = \exp\left(-\int_0^t \theta(s)\,ds\right).$$

Then the equation becomes

$$d(Z(t)\,Y(t,x)) = L(Z(t)\,Y(t,x))\,dt + Z(t)\,K(t,x)\,dt.$$

Hence, if we put

$$\widehat{Y} = Z(t)\,Y(t,x),$$

then $\widehat{Y}$ solves the equation

$$\begin{cases} d\widehat{Y}(t,x) &= L\widehat{Y}dt + Z(t)\,K(t,x)\,dt, \\ \widehat{Y}(0,x) &= \xi(x), \end{cases}$$

and we are back to the previous case.
Finally, consider the SPDE

$$dY(t,x) = LY dt + K(t,x)\, dt + h(t)\, Y(t)\, dB(t), \tag{3.3}$$

where h is a given bounded deterministic function and $K(t,x)$ is stochastic and adapted, and $\mathbb{E}[\int_0^T \int_{\mathbb{R}^n} K^2(t,x) dt dx] < \infty$. We handle this case by using white noise calculus on the Hida space $(\mathcal{S})^*$ of stochastic distributions: We introduce *white noise* $W_t \in (\mathcal{S})^*$ defined by

$$W_t = \frac{d}{dt} B(t),$$

and then we see that equation (3.3) can be written

$$\frac{d}{dt} Y(t,x) = LY + K(t,x) + Y(t)\, h(t) \diamond W_t,$$

where $\diamond$ denotes Wick multiplication. We refer to e.g. [10] for more information about white noise calculus. If we Wick-multiply this equation by

$$Z_t := \exp^{\diamond}\left(- \int_0^t h(s)\, dB(s) \right),$$

where in general $\exp^{\diamond}(\phi) = \sum_{n=0}^{\infty} \frac{1}{n!} \phi^{\diamond n}; \phi \in (\mathcal{S})^*$ is the Wick exponential, we get

$$Z_t \diamond \frac{d}{dt} Y(t,x) = L(Y \diamond Z_t) + K \diamond Z_t + Y(t)\, h(t) \diamond W_t \diamond Z_t. \tag{3.4}$$

Now

$$\frac{d}{dt}(Z_t \diamond Y) = Z_t \diamond \frac{d}{dt} Y(t) - Y(t) \diamond Z_t \diamond h(t)\, W_t, \tag{3.5}$$

and hence (3.4) can be written as

$$\frac{d}{dt} \underbrace{(Z_t \diamond Y_t)}_{\widehat{Y}_t} = L \underbrace{(Z_t \diamond Y_t)}_{\widehat{Y}_t} + K(t,x) \diamond Z_t.$$

This has the same form as (3.1). Hence the solution $\widehat{Y}$ is

$$\widehat{Y}(t,x) = P_t \xi(x) + \int_0^t P_{t-s}(K(s,.))(x) \diamond Z_s ds.$$

Now we go back from $\widehat{Y}$ to Y and get the solution

$$\begin{aligned} Y(t,x) &= \widehat{Y}(t,x) \diamond \exp^{\diamond}\left(\int_0^t h(s)\, dB(s) \right) \\ &= P_t \xi(x) \diamond \exp^{\diamond}\left(\int_0^t h(s)\, dB(s) \right) \\ &+ \int_0^t P_{t-s}(K(s,.))(x) \diamond \exp^{\diamond}\left(\int_s^t h(r)\, dB(r) \right) ds. \end{aligned} \tag{3.6}$$

Note that

$$\exp^{\diamond}\left(\int_0^t h\,(s)\,dB\,(s)\right) = \exp\left(\int_0^t h\,(s)\,dB\,(s) - \frac{1}{2}\int_0^t h^2\,(s)\,ds\right) > 0.$$

Recall the Gjessing-Benth lemma (see [15], [8] or Theorem 2.10.6 in [16] or Proposition 13 in [7]), which states that

$$\phi \diamond \exp^{\diamond}\left(\int_0^t h\,(s)\,dB\,(s)\right) = (\tau_{-h}\phi)\exp^{\diamond}\left(\int_0^t h\,(s)\,dB\,(s)\right),$$

where, for $\phi : \Omega \mapsto \mathbb{R}$, we define $\tau_{-h}\phi(\omega) = \phi(\omega - h); \omega \in \Omega$ to be the shift operator on Ω.

Using this in (3.6) we conclude that if

$$\xi \geq 0 \text{ and } K \geq 0 \text{ then } Y \geq 0.$$

We summarize what we have proved as follows:

Theorem 3.1 *Assume that* $\xi \in \mathcal{Y}^{(T)}$ *is deterministic,* $\mathbb{E}[\int_0^T \int_{\mathbb{R}^n} K^2(t,x)dtdx] < \infty$ *and let* $h : [0,T] \mapsto [0,T]$ *be bounded deterministic.*

1. *Then the unique solution* $Y(t,x) \in \mathcal{Y}^{(T)}$ *of the non-homogeneous SPDE*

$$\begin{aligned} dY\,(t,x) &= LY dt + K\,(t,x)\,dt + h\,(t)\,Y\,(t)\,dB\,(t)\,, \\ Y(0,x) &= \xi(x); \quad x \in \mathbb{R}^n \end{aligned}$$

is given by

$$\begin{aligned} Y\,(t,x) &= (\tau_{-h}P_t\xi)(x)\exp^{\diamond}\left(\int_0^t h\,(s)\,dB\,(s)\right) \\ &\quad + \int_0^t (\tau_{-h}P_{t-s}(K(s,.))(x)\exp^{\diamond}(\int_s^t h(r)dB(r))ds, \end{aligned}$$

where $\exp^{\diamond}(\int_s^t h(r)dB(r)) = \exp(\int_s^t h(r)dB(r) - \frac{1}{2}\int_s^t h^2(r)dr); \quad 0 \leq s \leq t \leq T.$

2. *In particular, if* $\xi(x) \geq 0$ *and* $K(t,x) \geq 0$ *for all* $(t,x) \in [0,T] \times \mathbb{R}^n$*, then* $Y(t,x) \geq 0$ *for all* $(t,x) \in [0,T] \times \mathbb{R}^n$*.*

Combining this with Theorem 2.1 we get

Theorem 3.2 *(Positivity) Assume that* $\xi \in \mathcal{Y}^{(T)}$ *is deterministic and let* $h : [0,T] \mapsto \mathbb{R}$ *be bounded and deterministic. Let* $Y(t,x) \in \mathcal{Y}^{(T)}$ *be the unique solution of the following SPDE with space interactions:*

$$Y(t,x) = \xi(x) + \int_0^t LY(s,x)ds + \int_0^t \overline{Y}(s,x)ds + \int_0^t h(s)Y(s,x)dB(s); \quad t \in [0,T], \quad (3.7)$$

given by Theorem 3.1.
Then if $\xi(x) \geq 0$ *for all* $x \in \mathbb{R}^n$*, we have* $Y(t,x) \geq 0$ *for all* $(t,x) \in [0,T] \times \mathbb{R}^n$*.*

4 Application to vaccine optimisation

Assume that the *density* $Y(t,x)$ *of infected individuals in a population* in a random/noisy environment changes over time t and space point x according to the following space-interaction reaction-diffusion equation

$$\begin{cases} dY(t,x) &= \frac{1}{2}\Delta Y(t,x)dt + \Big(b_0\overline{Y}(t,x) - u(t,x)Y(t,x)\Big)dt + \sigma_0 Y(t,x)dB(t), \\ Y(0,x) &= \xi(x) \geq 0; \quad x \in D, \\ Y(t,x) &= \eta(t,x) \geq 0; \quad (t,x) \in (0,T) \times \partial D, \end{cases}$$

where b_0, σ_0 are given constants modelling the effect on the growth $dY(t,x)$ of the term $\overline{Y}$ and of the noise, respectively, and $\overline{Y}(t,x) = G(x, Y(t,\cdot))$, where, as before, G is a space-averaging operator of the form

$$G(x,\varphi) = \frac{1}{V(K_\theta)}\int_{K_\theta} \varphi(x+y)dy; \quad \varphi \in L^2(D),$$

with $V(\cdot)$ denoting Lebesgue volume and

$$K_\theta = \{y \in \mathbb{R}^n; |y| < \theta\}$$

is the ball of radius $\theta > 0$ in $\mathbb{R}^n$ centered at 0.
By a slight extension of Theorem 3.2 (see Remark **??**), we know that $Y(t,x) \geq 0$ for all t, x.
If $u(t,x)$ represents our vaccine effort rate at (t,x), we define the total expected cost $J(u)$ of the effort by

$$J(u) = \mathbb{E}\Big[\frac{\rho}{2}\int_D \int_0^T u(t,x)^2 Y(t,x)dtdx + \int_D h_0(x)Y(T,x)dx\Big],$$

where $\rho > 0$ is a constant, and $h_0(x) > 0$ is a bounded function. Here we may regard the first quadratic term as the cost of the vaccination effort, with unit price ρ, and the second term as the cost of having remaining infection at time T. In this case the Hamiltonian is

$$H(t,x,y,\overline{y},p,q) = (\alpha\overline{y} - uy)p + \beta yq + \frac{\rho}{2}u^2 y,$$

and the adjoint equation satisfies

$$\begin{cases} dp(t,x) = -\Big[\frac{1}{2}\Delta p(t,x) - u(t,x)p(t,x) + \overline{\nabla}_{\overline{y}} H(t,x) + \beta q(t,x) + \frac{\rho}{2}u^2(t,x)\Big]dt + q(t,x)dB(t), \\ p(T,x) = h_0(x); \quad x \in D \\ p(t,x) = 0; \quad (t,x) \in (0,T) \times \partial D, \end{cases} \tag{4.1}$$

where, by Example **??**, $\overline{\nabla}_{\overline{y}} H(t,x) = v_D(x)\alpha p(t,x)$, with $v_D(x) := \frac{V((x+K_r)\cap D)}{V(K_r)}$.
The first order condition for an optimal $u = \widehat{u}$ for H together with the requirement that $Y(t,x) > 0$. lead to

$$\widehat{u}(t,x) = \frac{p(t,x)}{\rho}.$$

Hence the pair of random fields $(\widehat{p}, \widehat{q})$ becomes

$$\begin{cases} d\widehat{p}(t,x) & = -\Big[\frac{1}{2}\Delta\widehat{p}(t,x) + \frac{1}{2\rho}\widehat{p}^2(t,x) + v_D(x)\alpha\widehat{p}(t,x) + \beta\widehat{q}(t,x)\Big]dt + \widehat{q}(t,x)dB(t), \\ \widehat{p}(T,x) & = h_0(x); \quad x \in D, \\ \widehat{p}(t,x) & = 0; \quad (t,x) \in (0,T) \times \partial D. \end{cases} \tag{4.2}$$

Since h_0 and all the coefficients of this equation are deterministic, we can conclude that $\widehat{q} = 0$ and (4.2) reduces to the deterministic partial differential equation

$$\begin{cases} \frac{\partial}{\partial t}\widehat{p}(t,x) & = -\Big[\frac{1}{2}\Delta\widehat{p}(t,x) + \frac{1}{2\rho}\widehat{p}^2(t,x) + v_D(x)\alpha\widehat{p}(t,x)\Big], \\ \widehat{p}(T,x) & = h_0(x); \quad x \in D, \\ \widehat{p}(t,x) & = 0; \quad (t,x) \in (0,T) \times \partial D. \end{cases}$$

This is a (deterministic) Fujita type backward quadratic reaction diffusion equation. We could also from the beginning have allowed $h_0(x)$ to be random and satisfy $\mathbb{E}\left[\int_D h_0^2(x)dx\right] < \infty$. Then the equation (4.2) would have become a nonlinear backward *stochastic* reaction-diffusion equation. We will not discuss this further here, but refer to Bandle, & Levine [2], Dalang et al [9] and Fujita [13] and the references therein for more information.

References

[1] Agram, N., Hilbert, A., & Øksendal, B. (2020). Singular control of SPDEs with space-mean dynamics. Mathematical Control & Related Fields, 10(2), 425.

[2] Bandle, & Levine, H. (1994). Fujita type phenomena for reaction-diffusion equations with convection like terms. Differential and Integral Equations, 7(5-6), 1169-1193.

[3] Bensoussan, A. (1983). Stochastic maximum principle for distributed parameter systems. J. Franklin Inst. 315, 387-406.

[4] Bensoussan, A. (1983). Maximum principle and dynamic programming approaches of the optimal control of partially observed diffusions. Stochastics: An International Journal of Probability and Stochastic Processes, 9(3), 169-222.

[5] Bensoussan, A. (1991). Stochastic maximum principle for systems with partial information and application to the separation principle. Applied Stochastic Analysis. Gordon and Breach, 157-172.

[6] Bensoussan, A. (2004). Stochastic Control of Partially Observable Systems. Cambridge University Press.

[7] Benth, F. E. (1997). On the positivity of the stochastic heat equation. Potential Analysis, 6(2), 127-148.

[8] Benth, F. E., & Gjessing, H. K. (1994). A non linear parabolic equation with noise: a reduction method.

[9] Dalang, R. C., Khoshnevisan, D., & Zhang, T. (2019). Global solutions to stochastic reaction-diffusion equations with super-linear drift and multiplicative noise. Annals of Probability, 47(1), 519-559.

[10] Di Nunno, G., Øksendal, B., & Proske, F. (2009). Malliavin Calculus for Lévy Processes with Applications to Finance. Springer.

[11] Du, K., Tang, S. & Zhang, Q. (2013). Wm, p-solution ($p \geq 2$) of linear degenerate backward stochastic partial differential equations in the whole space. Journal of Differential Equations, 254(7), 2877-2904.

[12] Fuhrman, M., Ying Hu, Y. & Tessitore, G. (2013). Stochastic maximum principle for optimal control of SPDEs. Applied Mathematics & Optimization 68.2, 181-217.

[13] Fujita, H. (1966). On the blowing up of solutions of the Cauchy problem for $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. IA Math., 13, 109-124.

[14] Gawarecki, L., & Mandrekar, V. (2010). Stochastic Differential Equations in Infinite Dimensions: with Applications to Stochastic Partial Differential Equations. Springer Science & Business Media.

[15] Gjessing, H. K. (1995). Wick calculus with applications to anticipating stochastic differential equations. Institut Mittag-Leffler.

[16] Holden, H., Øksendal, B., Ubøe, J. & Zhang, T. (2010). Stochastic Partial Differential Equations. A Modelling, White Noise Functional Approach. Second Edition. Springer.

[17] Hu, Y., Ma, J., & Yong, J. (2002). On semi-linear degenerate backward stochastic partial differential equations. Probability Theory and Related Fields, 123(3), 381-411.

[18] Hu, Y., & Peng, S. (1990). Maximum principle for semilinear stochastic evolution control systems. Stochastics and Stochastic Reports, 33(3-4), 159-180.

[19] Hairer, M. (2009). An introduction to stochastic PDEs. arXiv preprint arXiv:0907.4178.

[20] Ma, J., & Yong, J. (1999). On linear, degenerate backward stochastic partial differential equations. Probability Theory and Related Fields, 113(2), 135-170.

[21] Marinelli, C., & Röckner, M. (2016). On the maximal inequalities of Burkholder, Davis and Gundy. Expositiones Mathematicae 34 (1), 1-26.

[22] Øksendal, B. (2005). Optimal control of stochastic partial differential equations. Stochastic Analysis and Applications, 23(1), 165-179.

[23] Øksendal, B., Proske, F., & Zhang, T. (2005). Backward stochastic partial differential equations with jumps and application to optimal control of random jump fields. Stochastics 77(5), 381-399.

[24] Øksendal, B., Sulem, A., & Zhang, T. (2014). Singular control and optimal stopping of SPDEs, and backward SPDEs with reflection. Mathematics of Operations Research, 39(2), 464-486.

[25] Øksendal, B., & Sulem, A.(2019). Applied Stochastic Control of Jump Diffusions. Third Edition. Springer.

[26] Pardouxt, É. (1980). Stochastic partial differential equations and filtering of diffusion processes. Stochastics, 3(1-4), 127-167.

[27] Pardoux, É. (1991). Filtrage non linéaire et équations aux dérivées partielles stochastiques associées. In Ecole d'Eté de Probabilités de Saint-Flour XIX—1989 (pp. 68-163). Springer.

[28] Da Prato, G., & Zabczyk, J. (2014). Stochastic Equations in Infinite Dimensions. Cambridge University Press.

[29] Prévôt, C., & Röckner, M. (2007). A Concise Course on Stochastic Partial Differential Equations. Springer.

[30] Röckner, M., & Zhang, T. (2007). Stochastic evolution equations of jump type: existence, uniqueness and large deviation principles. Potential Analysis, 26(3), 255-279.

[31] Smoller, J. (2012). Shock waves and reaction-diffusion equations (Vol. 258). Springer Science & Business Media.

[32] Wloka, J. (1987) Partial Differential Equations. Cambridge University Press.

[33] Zhou, X. Y. (1993). On the necessary conditions of optimal controls for stochastic partial differential equations. SIAM Journal on Control and Optimization, 31(6), 1462-1478.