EXISTENCE OF UNIQUE SOLUTION OF A FRACTIONAL WAVE EQUATION WITH FREE BOUNDARY CONDITIONS

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ABSTRACT
This paper investigates the problem of existence and uniqueness of solution under the traveling wave form for a free boundary problem of a space-fractional wave equation. It does so by applying the Banach's fixed point theorem.

1. INTRODUCTION

The partial differential equations (PDEs) of fractional order appear as a natural description of observed evolution phenomena in various scientific areas. The fractional derivative operators are non-local and this property is important in application because it allows to model the dynamics of many problems in physics, chemistry, engineering, medicine, economics, control theory, etc. For further reading on the subject, readers can refer to the following books (Samko et al. 1993 [15], Podlubny 1999 [13], Kilbas et al. 2006 [7], Diethelm 2010 [5]).

In this work, we shall give an example of a class of well-known fractional-order's PDEs; such the equation which is the space-fractional wave equation and is written as follows :

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^\alpha u}{\partial x^\alpha} + v(x,t)\,,\ c \in \mathbb{R}^*,\ 2 \le \alpha < 3, \tag{1}$$

where $u = u(x,t)$ and $v(x,t)$ are scalar functions of space variables $x \in [ct, X]$ and time $t \in [0,T]$, for $T > 0$ and $X > |c|\,T$. With

$$\frac{\partial^\alpha u}{\partial x^\alpha} = \begin{cases} \frac{\partial^m u}{\partial x^m}, & \alpha = m \in \mathbb{N}, \\ \int_{ct}^{x} \frac{(x-\tau)^{m-\alpha-1}}{\Gamma(m-\alpha)} \frac{\partial^m u(\tau,t)}{\partial \tau^m} d\tau, & m-1 < \alpha < m \in \mathbb{N}^*. \end{cases}$$

The space-fractional wave equation (1) becomes the wave equation for $\alpha = 2$ and $v = 0$, (see [6]).

The existence and uniqueness of solutions for fractional differential equations or fractional-order's PDEs have been investigated in recent years. For more on the subject, we refer the reader to the following works [1, 2, 3, 4, 7, 8, 9, 11, 12, 13, 16, 17, 18, 19].

Our main goal in this work is to determine the existence, uniqueness and main properties of solutions of the space-fractional PDE (1), under the traveling wave form :

$$u(x,t) = \exp\left(-c^2 t\right) f(x-ct)\,,\ \text{with } c \in \mathbb{R}^*, \tag{2}$$

the basic profile f is not known in advance and is to be identified.

We exemplify the role of Free Boundary Problems as an important source of ideas in modern analysis. With the help of a model problem, we illustrate the use of analytical techniques to obtain the existence and uniqueness of weak solutions via the use of the traveling wave method. This method permits us to reduce the fractional-order's PDE (1) to a fractional differential equation. This approach (2) is very promising and can also bring novel results for other applications in fractional-order's PDEs.

2. DEFINITIONS AND PRELIMINARY RESULTS

In this section, we present the necessary definitions from the fractional calculus theory. By $C([0,\lambda],\mathbb{R})$, we denote the Banach space of all continuous functions from $[0,\lambda]$ into $\mathbb{R}$ with the norm :

$$\|f\|_{\infty} = \sup_{\eta\in[0,\lambda]} |f(\eta)|.$$

We start with the definitions introduced in [7] with a slight modification in the notation.

Definition 1 ([7]) *The left-sided (arbitrary) fractional integral of order $\alpha > 0$ of a continuous function $f : [0,\lambda] \to \mathbb{R}$ is given by :*

$$\mathscr{I}_{0^+}^{\alpha} f(\eta) = \frac{1}{\Gamma(\alpha)} \int_0^{\eta} (\eta-\xi)^{\alpha-1} f(\xi)\, d\xi,\ \eta \in [0,\lambda].$$

$\Gamma(\alpha) = \int_0^{\infty} \xi^{\alpha-1} \exp(-\xi)\, d\xi$ *is the Euler gamma function.*

Definition 2 (Caputo fractional derivative [7]) *The left-sided Caputo fractional derivative of order $\alpha > 0$ of a function $f : [0,\lambda] \to \mathbb{R}$ is given by :*

$$^{C}\mathscr{D}_{0^+}^{\alpha} f(\eta) = \begin{cases} \frac{d^m f(\eta)}{d\eta^m}, \text{ for } \alpha = m \in \mathbb{N}, \\ \mathscr{I}_{0^+}^{m-\alpha} \frac{d^m f(\eta)}{d\eta^m} = \int_0^{\eta} \frac{(\eta-\xi)^{m-\alpha-1}}{\Gamma(m-\alpha)} \frac{d^m f(\xi)}{d\xi^m} d\xi, \text{ for } m-1 < \alpha < m. \end{cases} \tag{3}$$

Lemma 1 ([7]) *Assume that $^{C}\mathscr{D}_{0^+}^{\alpha} f \in C([0,\lambda],\mathbb{R})$, for all $\alpha > 0$, then :*

$$\mathscr{I}_{0^+}^{\alpha}\ {}^{C}\mathscr{D}_{0^+}^{\alpha} f(\eta) = f(\eta) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} \eta^k,\ m-1 < \alpha \leq m \in \mathbb{N}^*.$$

3. MAIN RESULTS

Throughout the rest of this paper, we have :

$$2 \leq \alpha < 3,\ T > 0 \text{ and } X > |c|\, T \text{ for some } c \in \mathbb{R}^*. \tag{4}$$

3.1. Statement of the free boundary problem and main theorems

In this part, we first attempt to find the equivalent approximate to the following free boundary problem of the space-fractional wave equation :

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^{\alpha} u}{\partial x^{\alpha}} + v(x,t), & (x,t) \in \Omega, \\ \frac{\partial u}{\partial x}(ct,t) = \frac{\partial^2 u}{\partial x^2}(ct,t) = 0,\ u(x,0) = f(x), & f \in C([0,\lambda],\mathbb{R}), \end{cases} \tag{5}$$

with $\Omega = [ct,X] \times [0,T]$, under the traveling wave form

$$u(x,t) = \exp\left(-c^2 t\right) f(\eta), \text{ with } \eta = x - ct\ . \tag{6}$$

Now, we give the principal theorem of this work.

Theorem 2 *Let $\alpha, c, T, X \in \mathbb{R}$, be the real constants given by (4) which satisfy the following inequality :*

$$0 < X + |c|\, T < \left(c^{-2}\Gamma(\alpha+1)\right)^{\frac{1}{\alpha}}.$$

If

$$\frac{\alpha (X+|c|\,T)^{\alpha} \left[2(X+|c|\,T)|c| + \alpha - 1\right]}{\Gamma(\alpha+1) - c^2 (X+|c|\,T)^{\alpha}} < (X+|c|\,T)^2, \tag{7}$$

then the problem (5) admits a unique solution in the traveling wave form (6).

3.2. Existence and uniqueness results of the basic profile

First, we should deduce the equation satisfied by the function f in (6) and used for the definition of traveling wave solutions.

Theorem 3 *Let $(x,t) \in [ct,X] \times [0,T]$, and $\varphi \in C([0,\lambda],\mathbb{R})$, be such that*

$$v(x,t) = c^2 \exp\left(-c^2 t\right) \varphi(\eta), \ \eta \in [0,\lambda],$$

for $\eta = x - ct$ and $\lambda = X + |c|T$, then the transformation (6) reduces the partial differential equation problem of space-fractional order (5) to the ordinary differential equation of fractional order of the form :

$$ {}^{C}\mathscr{D}^{\alpha}_{0^+} f(\eta) = c^2 f(\eta) + 2cf'(\eta) + f''(\eta) + \varphi(\eta), \ \eta \in [0,\lambda], \tag{8}$$

with the conditions :

$$f'(0) = f''(0) = 0, \tag{9}$$

Proof. The fractional equation resulting from the substitution of expression (6) in the original fractional-order's PDE (1), should be reduced to the standard bilinear functional equation (see [14]). First, for $\eta = x - ct$, we get $\eta \in [0,\lambda]$ and

$$\frac{\partial^2 u}{\partial t^2} = c^2 \exp\left(-c^2 t\right) \left[c^2 f(\eta) + 2cf'(\eta) + f''(\eta)\right]. \tag{10}$$

In another way, for $\xi = \tau - ct$, we get :

$$\begin{aligned}
\frac{\partial^\alpha u}{\partial x^\alpha} &= \frac{1}{\Gamma(3-\alpha)} \int_{ct}^{x} (\eta - \tau)^{2-\alpha} \frac{\partial^3 u(\tau,t)}{\partial \tau^3} d\tau \\
&= \frac{\exp\left(-c^2 t\right)}{\Gamma(3-\alpha)} \int_{ct}^{x} (x-\tau)^{2-\alpha} \frac{d^3}{d\tau^3} f(\tau - ct)\, d\tau \\
&= \frac{\exp\left(-c^2 t\right)}{\Gamma(3-\alpha)} \int_{0}^{\eta} (\eta - \xi)^{2-\alpha} \frac{d^3}{d\xi^3} f(\xi)\, d\xi \\
&= \exp\left(-c^2 t\right) {}^{C}\mathscr{D}^{\alpha}_{0^+} f(\eta). \qquad (11)
\end{aligned}$$

If we replace (10) and (11) in (1), we get :

$$ {}^{C}\mathscr{D}^{\alpha}_{0^+} f(\eta) = c^2 f(\eta) + 2cf'(\eta) + f''(\eta) + \varphi(\eta).$$

The proof is complete. ■

In what follows, we present some significant lemmas to show the principal theorems.

We have :

Lemma 4 *Let $f, f', f'', {}^{C}\mathscr{D}^{\alpha}_{0^+} f, \varphi \in C([0,\lambda],\mathbb{R})$, then the problem (8)–(9) is equivalent to the integral equation :*

$$\begin{aligned}
f(\eta) = f(0) + \frac{1}{\Gamma(\alpha)} \int_0^\eta (\eta - \xi)^{\alpha-1} \times \\
\left(c^2 f(\xi) + 2cf'(\xi) + f''(\xi) + \varphi(\xi)\right) d\xi. \qquad (12)
\end{aligned}$$

Proof. Let $f, f', f'', {}^{C}\mathscr{D}_{0^+}^{\alpha} f$, $\varphi \in C([0,\lambda],\mathbb{R})$, then by using Lemma 1, we reduce the fractional equation (8) to an equivalent fractional integral equation. By applying $\mathscr{I}_{0^+}^{\alpha}$ to the equation (8), we obtain :

$$\mathscr{I}_{0^+}^{\alpha}\ {}^{C}\mathscr{D}_{0^+}^{\alpha} f(\eta) = \mathscr{I}_{0^+}^{\alpha}\left(c^2 f(\eta) + 2cf'(\eta) + f''(\eta) + \varphi(\xi)\right). \tag{13}$$

From Lemma 1, we simply find :

$$\mathscr{I}_{0^+}^{\alpha}\ {}^{C}\mathscr{D}_{0^+}^{\alpha} f(\eta) = f(\eta) - f(0) - \eta f'(0) - \eta^2 f''(0).$$

by using (9), the fractional integral equation (13) gives us :

$$f(\eta) = \mathscr{I}_{0^+}^{\alpha}\left(c^2 f(\eta) + 2cf'(\eta) + f''(\eta) + \varphi(\eta)\right) + f(0).$$

The proof is complete. ∎

Lemma 5 *Let* $f, {}^{C}\mathscr{D}_{0^+}^{\alpha} f \in C([0,\lambda],\mathbb{R})$, *be such that* $f'(0) = f''(0) = 0$, *then* $\forall \eta \in [0,\lambda]$:

$$|2cf'(\eta)| + |f''(\eta)| \leq \frac{\lambda^{\alpha-2}(2\lambda|c| + \alpha - 1)}{\Gamma(\alpha)} \left\| {}^{C}\mathscr{D}_{0^+}^{\alpha} f \right\|_{\infty}. \tag{14}$$

Proof. By using Lemma 1, for all ${}^{C}\mathscr{D}_{0^+}^{\alpha} f \in C([0,\lambda],\mathbb{R})$, we get :

$$\begin{aligned} \mathscr{I}_{0^+}^{\alpha-1}\ {}^{C}\mathscr{D}_{0^+}^{\alpha} f(\eta) &= \frac{d}{d\eta} \mathscr{I}_{0^+}^{\alpha}\ {}^{C}\mathscr{D}_{0^+}^{\alpha} f(\eta) \\ &= f'(\eta) - f'(0) - 2f''(0)\eta. \end{aligned}$$

and

$$\begin{aligned} \mathscr{I}_{0^+}^{\alpha-2}\ {}^{C}\mathscr{D}_{0^+}^{\alpha} f(\eta) &= \frac{d^2}{d\eta^2} \mathscr{I}_{0^+}^{\alpha}\ {}^{C}\mathscr{D}_{0^+}^{\alpha} f(\eta) \\ &= f''(\eta) - 2f''(0). \end{aligned}$$

Moreover ; if $f'(0) = f''(0) = 0$, then :

$$\mathscr{I}_{0^+}^{\alpha-1}\ {}^{C}\mathscr{D}_{0^+}^{\alpha} f(\eta) = f'(\eta) \text{ and } \mathscr{I}_{0^+}^{\alpha-2}\ {}^{C}\mathscr{D}_{0^+}^{\alpha} f(\eta) = f''(\eta),$$

also, we have for any $\eta \in [0,\lambda]$,

$$\begin{aligned} |2cf'(\eta)| + |f''(\eta)| &= 2|c| \left| \mathscr{I}_{0^+}^{\alpha-1}\ {}^{C}\mathscr{D}_{0^+}^{\alpha} f(\eta) \right| + \left| \mathscr{I}_{0^+}^{\alpha-2}\ {}^{C}\mathscr{D}_{0^+}^{\alpha} f(\eta) \right| \\ &\leq \int_0^{\eta} \frac{2|c|(\eta-\xi)^{\alpha-2} \left| {}^{C}\mathscr{D}_{0^+}^{\alpha} f(\xi) \right|}{\Gamma(\alpha-1)} d\xi \\ &+ \int_0^{\eta} \frac{(\eta-\xi)^{\alpha-3} \left| {}^{C}\mathscr{D}_{0^+}^{\alpha} f(\xi) \right|}{\Gamma(\alpha-2)} d\xi \\ &\leq \frac{\lambda^{\alpha-2}(2\lambda|c| + \alpha - 1)}{\Gamma(\alpha)} \left\| {}^{C}\mathscr{D}_{0^+}^{\alpha} f \right\|_{\infty}. \end{aligned}$$

The proof is complete. ∎

Theorem 6 *If we put* $0 < \lambda < \left(c^{-2}\Gamma(\alpha+1)\right)^{\frac{1}{\alpha}}$ *and*

$$\frac{\alpha\lambda^{\alpha}(2\lambda|c| + \alpha - 1)}{\Gamma(\alpha+1) - c^2\lambda^{\alpha}} < \lambda^2, \tag{15}$$

then the problem (8)–(9) admits a unique solution on $[0,\lambda]$.

Proof. To begin the proof, we will transform the problem (8)–(9) into a fixed point problem $\mathscr{A}f(\eta)=f(\eta)$, with

$$\begin{aligned}\mathscr{A}f(\eta)=f(0)+\frac{1}{\Gamma(\alpha)}\int_0^\eta(\eta-\xi)^{\alpha-1}\times\\ \left(c^2f(\xi)+2cf'(\xi)+f''(\xi)+\varphi(\xi)\right)d\xi.\end{aligned}\tag{16}$$

We first notice that if $f,{}^C\mathscr{D}_{0^+}^\alpha f\in C([0,\lambda],\mathbb{R})$, then $\mathscr{A}f$ is being an operator of a polynomial and a primitive of continuous functions and its derivatives is indeed continuous (see (14) and the step 1 in this proof); therefore, it is an element of $C([0,\lambda],\mathbb{R})$, and is equipped with the standard norm :

$$\|\mathscr{A}f\|_\infty=\sup_{\eta\in[0,\lambda]}|\mathscr{A}f(\eta)|.$$

Let $f,g\in C([0,\lambda],\mathbb{R})$ be two functions that satisfy the problem (8)–(9), then

$$\begin{aligned}\mathscr{A}f(\eta)-\mathscr{A}g(\eta)=\frac{1}{\Gamma(\alpha)}\int_0^\eta(\eta-\xi)^{\alpha-1}\Big[c^2(f(\xi)-g(\xi))\\ +2c\left(f'(\xi)-g'(\xi)\right)+\left(f''(\xi)-g''(\xi)\right)\Big]d\xi.\end{aligned}$$

Also

$$|\mathscr{A}f(\eta)-\mathscr{A}g(\eta)|\le\int_0^\eta\frac{(\eta-\xi)^{\alpha-1}}{\Gamma(\alpha)}\left|{}^C\mathscr{D}_{0^+}^\alpha f(\xi)-{}^C\mathscr{D}_{0^+}^\alpha g(\xi)\right|d\xi.\tag{17}$$

For all $\eta\in[0,\lambda]$, we have :

$$\begin{aligned}\left|{}^C\mathscr{D}_{0^+}^\alpha f(\eta)-{}^C\mathscr{D}_{0^+}^\alpha g(\eta)\right|\le c^2|f(\eta)-g(\eta)|+2|c|\left|f'(\eta)-g'(\eta)\right|\\ +\left|f''(\eta)-g''(\eta)\right|.\end{aligned}$$

By using (14) from Lemma 5, we have :

$$\begin{aligned}\left\|{}^C\mathscr{D}_{0^+}^\alpha f-{}^C\mathscr{D}_{0^+}^\alpha g\right\|_\infty\le c^2\|f-g\|_\infty+\frac{\lambda^{\alpha-2}(2\lambda|c|+\alpha-1)}{\Gamma(\alpha)}\\ \times\left\|{}^C\mathscr{D}_{0^+}^\alpha f-{}^C\mathscr{D}_{0^+}^\alpha g\right\|_\infty.\end{aligned}$$

As $\Gamma(\alpha)-\lambda^{\alpha-2}(2\lambda|c|+\alpha-1)>0$, we have :

$$\left\|{}^C\mathscr{D}_{0^+}^\alpha f-{}^C\mathscr{D}_{0^+}^\alpha g\right\|_\infty\le\frac{c^2\Gamma(\alpha)}{\Gamma(\alpha)-\lambda^{\alpha-2}(2\lambda|c|+\alpha-1)}\|f-g\|_\infty.$$

From (17) we find :

$$\|\mathscr{A}f-\mathscr{A}g\|_\infty\le\frac{c^2\lambda^\alpha}{\Gamma(\alpha+1)-\alpha\lambda^{\alpha-2}(2\lambda|c|+\alpha-1)}\|f-g\|_\infty.$$

This implies that by (15), $\mathscr{A}$ is a contraction operator.

As a consequence Banach's contraction principle (see [10]), we deduce that $\mathscr{A}$ has a unique fixed point which is the unique solution of the problem (8)–(9) on $[0,\lambda]$. The proof is complete.
■

3.3. Proof of main theorems

In this part, we prove the existence and uniqueness of solutions of the following free boundary problem of the space-fractional wave equation :

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^\alpha u}{\partial x^\alpha} + v(x,t), \ (x,t) \in [ct, X] \times [0,T], & 2 \leq \alpha < 3, \ c \in \mathbb{R}^*, \\ \frac{\partial u}{\partial x}(ct,t) = \frac{\partial^2 u}{\partial x^2}(ct,t) = 0, \text{ and } u(x,0) = f(x), & f \in C([0,\lambda], \mathbb{R}), \end{cases} \tag{18}$$

under the traveling wave form :

$$u(x,t) = \exp\left(-c^2 t\right) f(\eta), \text{ with } \eta = x - ct. \tag{19}$$

Proof of Theorem 2

The transformation (19) reduces the problem of the higher order space-fractional wave equation (18) to the ordinary differential equation of fractional order of the form :

$$^C\mathscr{D}^\alpha_{0^+} f(\eta) = c^2 f(\eta) + 2cf'(\eta) + f''(\eta) + \varphi(\eta), \tag{20}$$

with the conditions :

$$f'(0) = f''(0) = 0. \tag{21}$$

Let $f \in C([0,\lambda], \mathbb{R})$ be a continuous function. By using Theorem 3, the condition (7) is equivalent to (15), which is :

$$\frac{\alpha\lambda^\alpha (2\lambda |c| + \alpha - 1)}{\Gamma(\alpha+1) - c^2\lambda^\alpha} < \lambda^2,$$

We already proved the existence of a single solution of the problem (20)–(21) in Theorem 6, provided that (15) hold space-fractional wave equation (18) under the traveling wave form (19). The proof is complete.

4. CONCLUSION

In this paper, we have discussed the existence of one solution for a class of space-fractional PDEs, which is known as a space-fractional wave equation with free boundary conditions, under the traveling wave form. The behavior of these solutions depends on some parameters that satisfy some conditions which make their existence in a finite time T. For that, we used the Banach contraction principle, while Caputo's fractional derivative is used as the differential operator.

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