ANISOTROPIC DEGENERATE PARABOLIC PROBLEMS IN $\mathbb{R}^N$ WITH VARIABLE EXPONENT AND LOCALLY INTEGRABLE DATA

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ABSTRACT

In this paper, we prove the existence and regularity of weak solutions for a class of nonlinear anisotropic parabolic equations in the whole $(0,T)\times\mathbb{R}^N$ with $p_i(x)$ growth conditions and locally integrable data. The functional setting involves Lebesgue-Sobolev spaces with variable exponents. Our results are generalizations of the corresponding results in the constant exponent case and some results given in Bendahmane and al. (Commun Pure Appl Anal 12 :1201–1220, 2013)

1. INTRODUCTION

Let us consider the following anisotropic parabolic problem :

$$(P)\qquad \begin{cases} \partial_t u - \sum_{i=1}^N D_i(d_i(t,x,u)a_i(t,x,Du)) + F(t,x,u) = f & \text{in } (0,T)\times\mathbb{R}^N \\ u(0,x) = u_0 & \text{on } \mathbb{R}^N \end{cases}$$

where $T>0$ a real number, $f\in L^1(0,T;L^1_{loc}(\mathbb{R}^N))$, $u_0\in L^1_{loc}(\mathbb{R}^N)$, $D_iu := \frac{\partial u}{\partial x_i}$. Suppose that $a_i : (0,T)\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$, $d_i : (0,T)\times\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ are Carathéodory functions and satisfying, a.e. $(t,x)\in(0,T)\times\mathbb{R}^N$, $\forall u\in\mathbb{R}$, $\forall\xi(\xi_1,\dots,\xi_N),\xi'(\xi'_1,\dots,\xi'_N)\in\mathbb{R}^N$, for all $i=1,\dots,N$, the following :

$$a_i(t,x,\xi)\xi_i \ge \beta|\xi_i|^{p_i(x)}, \tag{1.1}$$

$$|a_i(t,x,\xi)| \le \left(g(t,x) + h(t,x)\sum_{j=1}^N |\xi_j|^{p_j(x)}\right)^{1-\frac{1}{p_i(x)}}, \tag{1.2}$$

$$(a_i(t,x,\xi) - a_i(t,x,\xi'))(\xi_i - \xi'_i) > 0,\ \xi_i \ne \xi'_i, \tag{1.3}$$

$$\frac{c_2}{(1+|u|)^{\rho_i(x)}} \le d_i(t,x,u) \le c_1, \tag{1.4}$$

where β, c_1, c_2 are strictly positive real numbers, $\rho_i(x)\ge 0$, $g\in L^1(0,T;L^1_{loc}(\mathbb{R}^N))$, $h\in L^\infty_{loc}$ are a given positive functions, and the variable exponents $p_i : \mathbb{R}^N\to(1,\infty)$ are continuous functions. Let $F : (0,T)\times\mathbb{R}^N\times\mathbb{R}\to\mathbb{R}$ be a Carathéodory function satisfying the following conditions :

$$\sup_{|\sigma|\le\lambda} |F(t,x,\sigma)| \in L^1(0,T;L^1_{loc}(\mathbb{R}^N)), \quad \forall\lambda>0, \tag{1.5}$$

$$F(t,x,u)\,\mathrm{sign}(u) \ge \sum_{i=1}^N |u|^{s_i(x)},\ a.e.\ (t,x)\in(0,T)\times\mathbb{R}^N, \tag{1.6}$$

for all $u\in\mathbb{R}$, where $s_i(\cdot)>0, i=1,...,N$ are continuous functions on $\mathbb{R}^N$.
As a prototype example, we consider the model problem

$$(P_0)\qquad \begin{cases} \partial_t u - \sum_{i=1}^N \left(D_i\left(\frac{|D_iu|^{p_i(x)-2}D_iu}{(1+|u|)^{\rho_i(x)}}\right) - |u|^{s_i(x)-1}u\right) = f & \text{in} \quad (0,T)\times\mathbb{R}^N \\ u(0,x) = u_0 & \text{in} \quad \mathbb{R}^N \end{cases}$$

2. STATEMENTS OF RESULTS

Definition 2.1 *A function u is a weak solution of problem (P) if :*

$$u \in L^1(0,T;W^{1,1}_{loc}(\mathbb{R}^N)) \cap \left(L^{s_+(\cdot)}_{loc}((0,T)\times\mathbb{R}^N)\right),\, a_i \in L^1(0,T;L^1_{loc}(\mathbb{R}^N)),\ i=1,\dots,N$$

and

$$-\int_0^T\int_{\mathbb{R}^N} u\partial_t\varphi dxdt - \int_{\mathbb{R}^N}\varphi(0,x)u_0(x)dx + \sum_{i=1}^N\int_0^T\int_{\mathbb{R}^N} d_i(t,x,u)a_i(t,x,Du)D_i\varphi dxdt + \int_0^T\int_{\mathbb{R}^N} F(t,x,u)\varphi dxdt = \int_0^T\int_{\mathbb{R}^N}\varphi(t,x)f dxdt. \tag{2.1}$$

$\forall\varphi\in C^1_c([0,T)\times\mathbb{R}^N)$*, the C^1_c functions with compact support.*

Our main results are the following :

Theorem 2.2 *Let $f\in L^1(0,T;L^1_{loc}(\mathbb{R}^N))$, $\rho_i(x)=\rho$, assume that $p_i(\cdot), i=1,...,N$ are continuous functions such that for all $i=1,\dots,N$*

$$2+\frac{\rho N-1}{N+1} < p_i(\cdot) < \frac{\overline{p}(\cdot)(N+1)}{N(\rho+1)},\ \frac{1}{\overline{p}(\cdot)} = \frac{1}{N}\sum_{i=1}^N\frac{1}{p_i(\cdot)}, \tag{2.2}$$

$\overline{p}(\cdot)\le N+\frac{N(\rho+1)}{N+1}$*, and*

$$s_i(\cdot)\ge p_i(\cdot), \tag{2.3}$$

$$0\le\rho< p_i'(\cdot)-1. \tag{2.4}$$

Let a_i be a Carathéodory function satisfying (1.1)-(1.3) *and F satisfying* (1.5)-(1.6)*. Then, the problem (P) has at least one weak solution*

$$u\in\bigcap_{i=1}^N L^{q_i^-}\left(0,T;W^{1,q_i(\cdot)}_{loc}(\mathbb{R}^N)\right),$$

where $q_i(\cdot):\mathbb{R}^N\to[1,\infty)$ are continuous function such that

$$1\le q_i(\cdot)<\frac{p_i(\cdot)}{\overline{p}(\cdot)}\left(\overline{p}(\cdot)-\frac{N(\rho+1)}{N+1}\right),\quad \forall x\in\mathbb{R}^N. \tag{2.5}$$

Theorem 2.3 *Let $f\in L^1(0,T;L^1_{loc}(\mathbb{R}^N))$ and assume that $p_i(\cdot)>1$, $s_i(\cdot)>0$, $\rho_i(\cdot)\ge0$, $i=1,\dots,N$ are continuous functions on $\mathbb{R}^N$ such that*

$$s_i(\cdot)>(1+\rho_+(\cdot))(p_i(\cdot)-1),\quad \forall i=1,\dots,N. \tag{2.6}$$

$$p_i(\cdot)>1+\frac{1+\rho_i(\cdot)}{s_+(\cdot)}. \tag{2.7}$$

Let a_i be a function satisfying (1.1)-(1.3) *and F satisfy* (1.5)-(1.6)*. Then, the problem (P) has at least one weak solution*

$$u\in\bigcap_{i=1}^N L^{q_i^-}\left(0,T;W^{1,q_i(\cdot)}_{loc}(\mathbb{R}^N)\right),$$

where $q_i(\cdot):\mathbb{R}^N\to[1,\infty)$ are continuous functions such that

$$1\le q_i(\cdot)<\frac{p_i(\cdot)s_+(\cdot)}{1+s_+(\cdot)+\rho_i(\cdot)}. \tag{2.8}$$

3. CONCLUSIONS

In this paper, we have focused our attention on a class of nonlinear anisotropic parabolic equations with degenerate coercivity of the form

$$(P) \qquad \begin{cases} \partial_t u + A(u) + F(t,x,u) = f & \text{in } (0,T)\times\mathbb{R}^N \\ u(0,x) = u_0 & \text{on } \mathbb{R}^N \end{cases}$$

where the differential operator $A(u) = -\sum_{i=1}^{N} D_i(d_i(t,x,u)a_i(t,x,Du))$ and The right-hand f in $L^1(0,T;L^1_{loc}(\mathbb{R}^N))$, u_0 in $L^1_{loc}(\mathbb{R}^N)$. Suppose that $a_i : (0,T)\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$ be a Carathéodory function satisfying (1.1)-(1.3) and $d_i : (0,T)\times\mathbb{R}^N\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying :

$$\frac{c_2}{(1+|u|)^{\rho_i(x)}} \leq d_i(t,x,u) \leq c_1, \tag{3.1}$$

with $c_1, c_2 > 0$, and $\rho_i(x) \geq 0$ the differential operator A is not coercive if u is large.

Moreover, the lower order term $F : (0,T)\times\mathbb{R}^N\times\mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying (1.5)-(1.6).

In this work, we have been able to prove the following results :

— For $0 \leq \rho_i(x) = \rho < p_i'(.) - 1$, assume that $p_i(\cdot), i = 1,...,N$ are continuous functions such that for all $i = 1,\ldots,N$

$$2 + \frac{\rho N - 1}{N+1} < p_i(\cdot) < \frac{\overline{p}(\cdot)(N+1)}{N(\rho+1)},$$

and

$$s_i(\cdot) \geq p_i(\cdot),$$

we have established the existence and regularity of weak solutions to Problem (P) using local estimates for suitable approximate problems with variable exponent that have helped us to deal with the lower-order term as well as the regularity of the solution u and that's after passing the limit.

— For $\rho_i(\cdot) \geq 0$, $s_i(\cdot) > 0, p_i(\cdot) > 1$, $i = 1,\ldots,N$ are continuous functions on $\mathbb{R}^N$ such that

$$s_i(\cdot) > (1+\rho_+(\cdot))(p_i(\cdot)-1), \quad \forall i = 1,\ldots,N.$$

$$p_i(\cdot) > 1 + \frac{1+\rho_i(\cdot)}{s_+(\cdot)}.$$

we proved, some existence and regularity results for a weak solutions u in

$$\bigcap_{i=1}^{N} L^{q_i^-}\left(0,T;W^{1,q_i(\cdot)}_{loc}(\mathbb{R}^N)\right),$$

where

$$1 \leq q_i(\cdot) < \frac{p_i(\cdot)s_+(\cdot)}{1+s_+(\cdot)+\rho_i(\cdot)}.$$

We remind the reader that the same problem has been studied before and under certain conditions by each of

— F. Mokhtari, Nonlinear anisotropic parabolic equations in $\mathbb{R}^N$ with locally integrable data, 2013

The case $\rho_i(x) = 0, d_i = 1$ and the variable exponent $2 - \frac{1}{N+1} < p_i(x) = p_i < \frac{\overline{p}(N+1)}{N}$, the problem (P) has at least one weak solution $u \in \bigcap_{i=1}^{N} L^{q_i}\left(0,T;W^{1,q_i}_{loc}(\Omega)\right)$ where $1 \leq q_i < \frac{p_i}{\overline{p}}\left(\overline{p} - \frac{N}{N+1}\right), \quad \forall x \in \mathbb{R}^N$.

— M. Bendahmane, K. H. Karlsen, M. Saad, Nonlinear anisotropic elliptic and parabolic equations with variable exponents and L^1-data, 2013
The case $\rho_i(x) = 0, d_i = 1$ is in a bounded domain Ω, with $2 - \frac{1}{N+1} < p_i(x) < \frac{\overline{p}(.)(N+1)}{N}$, the problem (P) has at least one weak solution $u \in \bigcap_{i=1}^{N} L^{q_i^-}\left(0,T;W_0^{1,q_i(\cdot)}(\Omega)\right)$ where $1 \leq q_i(\cdot) < \frac{p_i(\cdot)}{\overline{p}(\cdot)}\left(\overline{p}(\cdot) - \frac{N}{N+1}\right), \quad \forall x \in \Omega.$